

**On the Relationship between Compressional Wave
Velocity of Saturated Porous Rocks and Density:
Theory and Application**

by

Marwah I. AL Ismail

Submitted to the Department of Earth, Atmospheric, and Planetary
Sciences

in partial fulfillment of the requirements for the degree of

Master of Science in Earth and Planetary Sciences

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2017



© Massachusetts Institute of Technology 2017. All rights reserved.

Signature redacted

Author
Department of Earth, Atmospheric, and Planetary Sciences
January 20, 2017

Signature redacted

Certified by.....
✓ Frank Dale Morgan
Professor of Geophysics
Thesis Supervisor

Signature redacted

Accepted by
Robert D. van der Hilst
Schlumberger Professor of Earth and Planetary Sciences Head,
Department of Earth, Atmospheric, and Planetary Sciences

On the Relationship between Compressional Wave Velocity of Saturated Porous Rocks and Density: Theory and Application

by

Marwah I. AL Ismail

Submitted to the Department of Earth, Atmospheric, and Planetary Sciences
on January 20, 2017, in partial fulfillment of the
requirements for the degree of
Master of Science in Earth and Planetary Sciences

Abstract

Understanding the velocity of the compressional waves travelling through rocks is essential for the purposes of applied geophysics in such areas as groundwater and hydrocarbon exploration. The wave velocity is defined theoretically by the Newton-Laplace equation, which relates the wave velocity, V , to the square root of the ratio of the rock's elastic modulus, M , and its density, ρ (Bourvie et al., 1987). Therefore, the equation indicates that the velocity is inversely proportional to density. However, the in-situ field measurements and laboratory experiments of compressional wave velocity through different rocks show otherwise. In other words, the velocity is directly proportional to approximately the 4th power of density as stated by Gardner (Gardner et al., 1974).

This thesis investigates the inconsistency between theory and observations regarding the relationship between velocity and density of saturated porous rocks. The inconsistency is clarified by deriving a new expression for the elastic modulus, M , using Wyllie's time average equation and the Newton-Laplace equation. The new derived expression of the elastic modulus, M , provides dependence of M on density to approximately the 9th power. In addition, Gardner's equation is modified to accurately obtain the velocity over the entire range of densities (from 1.00 g/cm^3 to around 3.00 g/cm^3) and porosity (from 0% to 100%).

The end of this thesis is an application of the previous outcomes with real data sets, where the results validate the derived expression of the elastic modulus as well as the generalized form of Gardner's equation.

Thesis Supervisor: Frank Dale Morgan
Title: Professor of Geophysics

Acknowledgments

I would like to take this opportunity to express my thanks to the people who supported, guided and helped me throughout my study at MIT. First, I would like to sincerely thank my thesis advisor Prof. Dale Morgan, who kept supporting and guiding me until the last minute. His guidance helped me in all the time of research and writing of this thesis. I will never forget his insightful, adventurous and motivational stories.

I also would like to thank my committee members Prof. Nafi Toksz, Prof. Brian Evans and Dr. Michael Fehler for their valuable comments and suggestions and for making the defense happen on a short notice.

I would like to thank my colleagues in the group, whom I call friends, who kept supporting me throughout my study and work on my research. They made this journey more enjoyable and memorable. I also thank my office mates, colleagues at ERL and MIT and the administrative staff for their support.

I also would like to thank Saudi Aramco Oil Company for the full scholarship and contribution by providing data that was useful for my research. I am also truly thankful for my mentor back in Saudi and my colleagues in the division, who encouraged and helped me during my study.

Special thanks to my family for their limitless love and guidance. They have always encouraged me towards excellence. In the end, no words are enough to convey my warmest thanks to my dear husband, Ahmed Alzayer, for his continuous love, support, patience and understanding throughout these years. I hope to make him and my family proud.

Contents

1	Introduction	17
1.1	The problem and objective	17
1.2	Thesis Outline	18
2	Literature Review	23
2.1	The compressional wave velocity in history	23
2.2	Theory of Elasticity	24
2.2.1	Hooke's Law	24
2.2.2	Young Modulus	25
2.2.3	Poisson's Ratio	25
2.2.4	Bulk Modulus	26
2.2.5	Shear Modulus	26
2.3	The derivation of the compressional wave velocity	26
2.4	Effective Media Theories	29
2.4.1	Voigt and Reuss Bounds	29
2.4.2	Hashin-Shtrikman Bounds	30
2.4.3	Hill Average Moduli Estimate	30
2.5	Wyllie's Equation (Time Average Equation)	31
3	Observations	33
3.1	Compressional Wave Velocity through Gaseous Elements	33
3.2	Compressional Wave Velocity through non-porous Solid Elements . .	36
3.3	Compressional Wave Velocity through Rocks	39

3.4	Conclusion	43
4	Methodology	45
4.1	Derivation of new expression for the elastic modulus, M	45
4.2	Applying the new expression to synthetic data	50
4.3	Comparing the elastic modulus using the new expression to the elastic modulus using the effective media theories and the geometric mean	53
4.4	Generalized Gardner's equation	59
5	Applications to Real Data	61
5.1	Introduction	61
5.2	Applying the new expression of the elastic modulus, M , to real data	61
5.3	Applying the generalized form of Gardner's equation to real data	66
6	Conclusion	93

List of Figures

1-1	A page from the book <i>Acoustics of Porous Media</i> that demonstrates the wave velocity equation (Bourvie et al., 1987)	20
1-2	A page from the book <i>Introduction to the Physics of Rocks</i> that demonstrates the wave velocity equation (Gue'guen and Palciauskas, 1994) .	21
1-3	A page from the book <i>Exploration Seismology</i> that demonstrates the wave velocity equation (Sheriff and Geldart, 1995)	22
2-1	An elastic object with applied stresses	27
3-1	Measured velocity versus density for gaseous elements	34
3-2	Log of measured velocity versus log of density for gaseous elements .	35
3-3	Measured velocity versus density for non-porous solid elements	36
3-4	Log of measured velocity versus log of density for non-porous solid elements	38
3-5	Measured velocity versus density for rocks	40
3-6	Log of measured velocity versus log of density for rocks	41
3-7	Gardner's log-log plot of velocity versus density through rocks (After (Gardner et al., 1974))	42
3-8	Merged log velocity versus log density for gases, solids and rocks . . .	43
4-1	The calculated velocity using Wyllie's equation versus density along with the best-fit line	47
4-2	The calculated velocity using Wyllie's equation versus porosity along with the best-fit line	48

4-3	The calculated elastic modulus using the new derived expression versus density	51
4-4	The calculated elastic modulus using the new derived expression versus porosity	52
4-5	The calculated elastic modulus using the new derived expression, calculated elastic modulus using $[M = (a\rho^b + c)^2\rho, rmse = 0.05]$ and the calculated elastic modulus using $[M = a\rho^b, rmse = 0.67]$ versus density	54
4-6	The calculated effective elastic modulus using: the new derived expression, effective media theories and the geometric mean versus density .	55
4-7	The calculated effective elastic modulus using: the new derived expression, effective media theories and the geometric mean versus porosity	56
4-8	The calculated velocity using the elastic modulus calculated by: the new derived expression, effective media theories and the geometric mean versus density	57
4-9	The calculated velocity using: the new derived expression, effective media theories and the geometric mean versus porosity	58
4-10	The calculated velocity using Gardner's equation versus density . . .	60
5-1	The velocity versus density for well No. 1	62
5-2	The velocity versus density for well No. 2	63
5-3	The velocity versus density for well No. 3	64
5-4	The velocity versus density for well No. 4	65
5-5	The calculated velocity using the new derived expression of the elastic modulus, M, versus measured velocity for Limestone within Well No. 1 ($rmse = 0.03$)	67
5-6	The calculated velocity using the new derived expression of the elastic modulus, M, and measured velocity versus density for Limestone within Well No. 1 ($rmse = 0.03$)	68

5-7	The calculated velocity using the new derived expression of the elastic modulus, M, versus measured velocity for Dolomite within Well No. 1 (<i>rmse</i> = 0.05)	69
5-8	The calculated velocity using the new derived expression of the elastic modulus, M, and measured velocity versus density for Dolomite within Well No. 1 (<i>rmse</i> = 0.05)	70
5-9	The calculated velocity using the new derived expression of the elastic modulus, M, versus measured velocity for Limestone within Well No. 2 (<i>rmse</i> = 0.06)	71
5-10	The calculated velocity using the new derived expression of the elastic modulus, M, and measured velocity versus density for Limestone within Well No. 2 (<i>rmse</i> = 0.06)	72
5-11	The calculated velocity using the new derived expression of the elastic modulus, M, versus measured velocity for Limestone within Well No. 3 (<i>rmse</i> = 0.07)	73
5-12	The calculated velocity using the new derived expression of the elastic modulus, M, and measured velocity versus density for Limestone within Well No. 3 (<i>rmse</i> = 0.07)	74
5-13	The calculated velocity using the new derived expression of the elastic modulus, M, versus measured velocity for Dolomite within Well No. 3 (<i>rmse</i> = 0.10)	75
5-14	The calculated velocity using the new derived expression of the elastic modulus, M, and measured velocity versus density for Dolomite within Well No. 3 (<i>rmse</i> = 0.10)	76
5-15	The calculated velocity using the new derived expression of the elastic modulus, M, versus measured velocity for Clean Sands within Well No. 4 (<i>rmse</i> = 0.05)	77
5-16	The calculated velocity using the new derived expression of the elastic modulus, M, and measured velocity versus density for Clean Sands within Well No. 4 (<i>rmse</i> = 0.05)	78

5-17	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Limestone within Well No. 1 ($rmse = 0.03$)	80
5-18	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Limestone within Well No. 1 ($rmse = 0.03$)	81
5-19	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Dolomite within Well No. 1 ($rmse = 0.05$)	82
5-20	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Dolomite within Well No. 1 ($rmse = 0.05$)	83
5-21	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Limestone within Well No. 2 ($rmse = 0.05$)	84
5-22	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Limestone within Well No. 2 ($rmse = 0.05$)	85
5-23	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Limestone within Well No. 3 ($rmse = 0.05$)	86
5-24	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Limestone within Well No. 3 ($rmse = 0.05$)	87
5-25	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Dolomite within Well No. 3 ($rmse = 0.06$)	88
5-26	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Dolomite within Well No. 3 ($rmse = 0.06$)	89

5-27	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Clean Sands within Well No. 4 ($rmse = 0.05$)	90
5-28	The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Clean Sands within Well No. 4 ($rmse = 0.05$)	91
6-1	Linking the Newton-Laplace equation to Wyllie's equation to produce new expression for the elastic modulus, M, the approximate relationship between the elastic modulus and density, and the approximate relationship between velocity and density	94

List of Tables

3.1	The measured compressional wave velocities and densities for gases	34
3.2	The measured compressional wave velocities and densities for non-porous solids	37
3.3	The average measured compressional wave velocities and average densities for rocks	39
4.1	The parameters used to calculate the velocity using Wyllie's equation	46
5.1	The average of the parameters used in calculating the velocity using the new derived expression of the elastic modulus	66
5.2	The calculated normalized <i>rmse</i> between the calculated velocity using the new expression of M and the measured velocity for each lithology interval	67
5.3	The fit parameters "a" and "b" and the normalized <i>rmse</i> for each lithology	79

Chapter 1

Introduction

1.1 The problem and objective

Many scientists throughout history spent a tremendous amount of time studying the compressional wave behavior through different mediums. Understanding the velocity of waves provides important information about the properties of the medium in which the wave is travelling. In exploration geophysics for water and hydrocarbons, for example, seismic wave velocity is one of the key parameters that has to be accurately measured.

The wave velocity is related to the elastic properties of the rock and its density by the theoretical expression (Bourvie et al., 1987):

$$V = \sqrt{\frac{M}{\rho}} \quad (1.1)$$

where V is the wave velocity, M is the elastic modulus and ρ is the density. The equation, which is known as the Newton-Laplace equation, relates the compressional wave velocity directly to the square root of the elastic modulus and inversely to the square root of density. However, the latter dependence opposes the real observed relationship between velocity and density. The experimental and laboratory measurements on rocks show that the compressional wave velocity actually increases with increasing density (Gardner et al., 1974).

It is important to note that most of the books that describe this relationship for rocks don't demonstrate this contradiction. An example is seen in figure 1-1, which is a page from the book *Acoustics of Porous Media* (Bourvie et al., 1987), that has the expression of the wave velocity equation without any further clarification about the real relationship between velocity and density. Another example is in figure 1-2 from the book *Introduction to the Physics of Rocks* (Gue'guen and Palciauskas, 1994) where any demonstration is missing. On the other hand, the figure 1-3 is a page from the book *Exploration Seismology* (Sheriff and Geldart, 1995) where the authors do notify the readers about the interrelation between the elastic modulus and density. However, the latter still does not demonstrate that the velocity increases with increasing density. The readers will assume that the dependence of velocity on density is what is seen in the equation rather than the true relationship, which creates a misunderstanding of the real physical behaviour of the velocity.

The current reasoning for this inconsistency between the theory and observation is explained by the fact that the elastic modulus is also dependent on density. In other words, the velocity is a function of the elastic modulus and density and the elastic modulus itself is also a function of density, i.e. $V = f(M, \rho)$ and $M = g(\rho)$. This thesis investigates the relationship between the compressional wave velocity and density for rocks as well as the relationship between the elastic modulus and density. The results of this investigation attempts to clarify and quantify the dependence of velocity and elastic modulus on density. In addition, the results are applied using real data to validate the findings.

1.2 Thesis Outline

The second chapter of this thesis is a review of the history of the wave velocity equation and the theory of elasticity. Chapter 3 shows real observations of the compressional wave velocity through different mediums such as gases, solids and rocks. The conclusion of chapter 3 directs the research toward deriving a new expression of the elastic modulus, M , that results in a remarkable relationship between M and

density. Chapter 5 shows the results of applying the findings of chapter 4 to real data. Chapter 6 highlights the main conclusion points of this thesis.

extent of fracture porosity by the increase in velocity with pressure. The ordinate in the diagram represents the normalized difference between the compressional velocity without pressure and that under a pressure of 1 GPa for dry samples: the greater the number of cracks, the more the velocity varies with pressure.

As already noted, the Pierre shale samples used in Fig. 5.2 display a different velocity function than Fig. 5.1. The velocity plateau as a function of pressure, which represents the closure of its "last" crack for the sample concerned, is not observed for this rock at 120 MPa. In fact, Jones and Wang (1981) observed this phenomenon continuing up to 0.4 GPa. It is possible that the increase in velocity as a function of pressure shows, in addition to the continuous closure of the microcracks, an alignment of the clay crystals in the minimum shear strength plane (Tosaya, 1982). Chalk, the second example in Fig. 5.2, is very complicated because the structure of the material varies according to the pressure applied (creep). We shall not go into further detail for either Pierre shale or chalk, assuming that the variation in velocity versus pressure are only important if the structure of the material investigated is not changed irreversibly by the experiment. For clays as well as chalk, there is good reason to suspect hysteresis in the curve of velocity vs. effective pressure.

We have shown that the increase in velocity with pressure results from the closure of the cracks, and that this closure is reflected by a greater rigidity of the material under pressure (i.e. an increase in the corresponding elastic modulus). In fact, it must be remembered that any velocity V can be expressed in the form:

$$V = \sqrt{\frac{M}{\rho}} \quad (5.1)$$

where M is the elastic modulus, and ρ the density, and that, consequently, at constant ρ , an increase in elastic modulus implies a rise in velocity. The behavior of cracks and pores under confining pressure was modeled by Walsh (1965, 1969) and Wu (1966) for a pore or crack included in a matrix. The equations satisfied by K , the bulk modulus, and μ , the shear modulus of the rock for the dry sample are the following:

$$\frac{1}{K} = \frac{1}{K_1} \left(1 + A \frac{\phi}{e} \right) \quad (5.2)$$

$$\frac{1}{\mu} = \frac{1}{\mu_1} \left(1 + B \frac{\phi}{e} \right) \quad (5.3)$$

where K_1 and μ_1 are the solid moduli, e the aspect ratio of the pore or crack ($e = 1$ for a sphere, $e \ll 1$ for a crack), ϕ the porosity, and A and B are constants depending on the characteristics of the medium and close to 1.

For the saturated sample:

$$\frac{1}{K} \approx \frac{1}{K_1} (1 + A' \phi) \quad (5.4)$$

$$\frac{1}{\mu} \approx \frac{1}{\mu_1} \left(1 + B' \frac{\phi}{e} \right) \quad (5.5)$$

where A' and B' are constants depending on K_1 and μ_1 (and fluid bulk modulus for A') and close to 1. It can be seen that, if $e = 1$, the effect of the pore on the moduli is negligible

Figure 1-1: A page from the book *Acoustics of Porous Media* that demonstrates the wave velocity equation (Bourvie et al., 1987)

VII. Acoustic Properties

BECAUSE of the important role of seismic prospecting and sonic logging, understanding the acoustic rock properties (velocity and attenuation) is essential for these and other geophysical applications. We have seen in chapter IV that the acoustic velocities V_p and V_s can be expressed simply in terms of the elastic constants and density of a medium. The elastic constants and density depend on lithology, porosity, fluid saturation, pressure, and temperature. Attenuation is an independent parameter which is potentially rich in information, but the many complex processes of attenuation are much more difficult to interpret. As a general rule, elastic anisotropy is not very pronounced, but there are certain cases where it must be taken into account.

VELOCITY OF ELASTIC WAVES

Velocity and Lithology

Lithology is an important factor which affects the velocities V_p and V_s . In *sedimentary rocks*, on the average, higher velocities are observed in carbonates than in sandstones and marl (fig. VII.1). But, unless we know their exact composition and/or their porosity, sandstone and calcite cannot be distinguished simply on the basis of V_p . Let us assume for the time being that the rocks are isotropic, homogeneous, and elastic. Then utilizing equations IV.8 and 9:

$$V_p = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}} \quad V_s = \sqrt{\frac{\mu}{\rho}}$$

The ratio V_p/V_s is interesting from the point of view of lithology because it only depends on the Poisson's ratio ν . With the aid of table IV.2, one can show

$$\nu = \frac{1}{2} \frac{(V_p/V_s)^2 - 2}{(V_p/V_s)^2 - 1} \quad \text{VII.1}$$

Figure VII.2 shows that the V_p/V_s ratio can be used to differentiate between sandstone and limestone. This result is partially a consequence

Figure 1-2: A page from the book *Introduction to the Physics of Rocks* that demonstrates the wave velocity equation (Gue'guen and Palciauskas, 1994)

5 Seismic velocity

Overview

Knowledge of velocity values is essential in determining the depth, dip, and horizontal location of reflectors and refractors, in determining whether certain things like head waves and velocity distortions occur, and in ascertaining the nature of rocks and their interstitial fluids from velocity measurements.

We develop a heuristic appreciation of the factors that affect seismic velocity from a conceptual model of a sedimentary rock. F. Gassmann, M. A. Biot, and J. Geertsma developed a model for a fluid-filled porous rock, and G. H. P. Gardner, L. W. Gardner, and A. R. Gregory hypothesized that microcracks in non-porous rocks lower velocity. Fracturing also generally lowers velocity.

Lithology is the most obvious factor we would expect to control velocity. However, velocity ranges are so broad and there is so much overlap that velocity alone does not provide a good basis for distinguishing lithology. Sand velocities, for example, can be smaller or larger than shale velocities, and the same is true for densities; both velocity and density play important roles in seismic reflectivity.

Porosity appears to be the most important single factor in determining a rock's velocity, and the dependence of porosity on depth of burial and pressure relationships makes velocity sensitive to these factors also. Velocity is generally lowered when gas or oil replaces water as the interstitial fluid, sometimes by so much that amplitude anomalies result from hydrocarbon accumulations.

The near-surface layer of the earth usually differs markedly from the remainder of the earth in velocity and some other properties. This makes the near-surface low-velocity layer (LVL) especially important; our determinations of depths, attitudes, and continuity of deeper events are affected as reflections pass through this layer. In arctic areas, a zone of permanently frozen earth, permafrost, distorts deeper events because of an exceptionally high velocity. Fluid pressure that exceeds that of a column of fluid extending to the surface ("normal" pressure) lowers seismic velocity; this is used to predict abnormal pressures. Gas hydrates that form in the sediments just below the sea floor in deep water also produce velocity changes.

Velocity terminology is often misused and causes much confusion. Section 5.4.1 attempts to clarify the precise meaning of average, root-mean-square, stack-

ing, interval, Dix, phase, group, apparent, and other velocity terms.

Seismic velocity is measured in boreholes by sonic logs (§5.4.3) (and by vertical seismic profiling discussed in §13.4). Velocity is also measured by surface seismic data because of the dependence of normal moveout on velocity. The reflection-coefficient equation can be used to obtain velocity information from amplitudes, a form of inversion.

5.1 Model of a sedimentary rock

5.1.1 A pack of uniform spheres

Seismic velocity as given by eqs. (2.58) and (2.59) relates to a homogeneous medium, but sedimentary rocks are far from homogeneous. These equations can be written, for solid media,

$$\alpha^2 = (\lambda + 2\mu)/\rho, \quad \beta^2 = \mu/\rho,$$

and for fluid media,

$$\alpha^2 = \lambda/\rho, \quad \beta^2 = 0,$$

hence, in general,

$$V = (K/\rho)^{1/2}, \quad (5.1)$$

where K is the effective elastic parameter. Thus, the dependence of V upon the elastic constants and density appears to be straightforward. In fact, the situation is much more complicated because K and ρ are interrelated, both depending to a greater or lesser degree upon the material and structure of the rock, the lithology, porosity, interstitial fluids, pressure, depth, cementation, degree of compaction, and so on. The most notable inhomogeneity of sedimentary rocks is that they are porous, containing fluid-filled spaces within them. *Porosity* is simply the pore volume per unit volume. Wang and Nur (1992b) discuss theories relating seismic velocity to the composition of rocks.

The simplest rock model consists of identical spheres arranged in a cubic pattern (fig. 5.1a) with the matrix subjected to a compressive pressure \mathcal{P} . If the radius of the spheres is R , the force F pressing two adjacent spheres together is the total force acting on a layer of $n \times n$ spheres (that is, $(2Rn)^2\mathcal{P}$) divided by the number of spheres (n^2), or $F = 4R^2\mathcal{P}$. This force causes a point of contact to become a circle of contact

Figure 1-3: A page from the book *Exploration Seismology* that demonstrates the wave velocity equation (Sheriff and Geldart, 1995)

Chapter 2

Literature Review

2.1 The compressional wave velocity in history

The velocity of compressional waves has been a major subject of investigation throughout the history (Gassman, 1951; Biot, 1956b,a; Geertsma and Smit, 1961; Domenico, 1977). Many scientists tried to measure and derive an expression that would provide an accurate value for the velocity of sound, which is a compressional wave, in air. It started in the 16th century when Isaac Newton derived a theoretical expression to calculate the velocity of sound using the pressure and the density of the air in which the sound wave is traveling (Finn, 1964), which is:

$$V = \sqrt{\frac{P}{\rho}} \quad (2.1)$$

where V is the velocity of sound wave, P is the pressure of air and ρ is the density of air. During the same time, other scientists were conducting experiments to measure the speed of sound through air. They found that the measured experimental value was almost 20% higher than the theoretical one, and no one was able to explain this excess over theory (Finn, 1964).

In the 18th century, Pierre-Simon Laplace investigated the speed of sound problem and was able to explain the inconsistency between the theoretical and experimental values. He found out that Newton followed Boyle's law of heat dissipation in his

derivation; i.e., the temperature of the medium does not change throughout the process of wave propagation (compression and rarefaction), making it an isothermal process. Conversely, Laplace identified that when the sound wave propagates, the air particles oscillate in high speed that would not allow the heat to escape. Thus, the temperature of the medium is raised and, as a result, the pressure is raised. This process is known as an adiabatic process (Finn, 1964; de Podesta, 2002).

To account for this new understanding of the problem, Laplace related the speed of sound to the square root of Gamma, γ , which is the ratio of the principle heat capacities¹ (Finn, 1964):

$$\gamma = \frac{C_P}{C_V} \quad (2.2)$$

where C_P is the heat capacity at constant pressure and C_V is the heat capacity at constant volume. Thus, the new equation for the speed of sound, known thereafter as The Newton-Laplace Equation, becomes:

$$V = \sqrt{\frac{K}{\rho}} \quad (2.3)$$

where $K = \gamma P$ and it is defined as the Bulk modulus for gases. Using the above equation along with $\gamma = 1.42$ for air gives a value of 331.2 m/s (Finn, 1964; de Podesta, 2002), which is in great agreement with the experimental measurements.

2.2 Theory of Elasticity

2.2.1 Hooke's Law

One fundamental law in the theory of elasticity is Hooke's Law, which addresses the deformation of bodies under load (Jones, 2009). Hooke's law is expressed by:

$$F = k\Delta d \quad (2.4)$$

¹Heat capacity is the ratio of heat added to or removed from a gas to the subsequent change in temperature.

where F is the force, Δd is the resulting length of elongation or compression, and k is a constant that characterizes the material, which was a spring in Hooke's experiment.

In addition, Robert M. John stated in his book *Deformation Theory of Plasticity* (Jones, 2009) that the first one to establish a linear relationship between stress and strain is James Bernoulli in 1705 (Jones, 2009). John, in his book, states that Bernoulli saw that a relation giving the ratio (force)/(area), or mean stress, as a function of strain characterizes the material (Jones, 2009).

2.2.2 Young Modulus

Leonhard Euler worked along with John Bernoulli on the Bernoulli-Euler bending theory. In his work on the theory, he characterized the material by its modulus of extension, which is what is known today as Young modulus. Clearly, the modulus is mistakenly referred to Thomas Young who came later and worked toward defining a limit to the linear relation between stress and strain rather than defining the relationship itself (Jones, 2009).

Euler Modulus, E , is defined as the constant of proportionality between extensional stress and extensional strain, in a uniaxial stress system (Mavko et al., 2003; Sheriff and Geldart, 1995):

$$E = \frac{\sigma_x}{\varepsilon_x} \quad (2.5)$$

where σ_x is the stress in the x-direction and ε_x is the strain in the x-direction.

2.2.3 Poisson's Ratio

Simon Denis Poisson observed that there is a relationship between the strain in the perpendicular (transverse) direction to the applied stress and the strain in the direction of the applied stress (axial) (Jones, 2009). This relationship is known as Poisson's ratio (Mavko et al., 2003):

$$\nu = \frac{\varepsilon_{transverse}}{\varepsilon_{axial}} \quad (2.6)$$

2.2.4 Bulk Modulus

Bulk modulus, K , is the constant of proportionality between pressure, P , and volumetric strain, ε_V (Mavko et al., 2003):

$$K = \frac{P}{\varepsilon_V} \quad (2.7)$$

where P is the normal stress applied in the x-, y- and z-directions. ε_V , the volumetric strain, is the equivalent of the resultant change in volume relative to the original volume (Mavko et al., 2003; Sheriff and Geldart, 1995):

$$\varepsilon_V = \frac{\Delta V}{V} \quad (2.8)$$

2.2.5 Shear Modulus

Shear modulus, μ , is the constant of proportionality between an extensional stress and the resultant strain in the perpendicular direction, defined as (Mavko et al., 2003):

$$\mu = \frac{\sigma_x}{\varepsilon_y} \quad (2.9)$$

where σ_x is the stress in the x-direction and ε_y is the strain in the y-direction.

2.3 The derivation of the compressional wave velocity

The derivation of the wave equation requires the understanding of stress, strain and their relation to each other (White, 1965; Sheriff and Geldart, 1995).

Any elastic object that experiences a stress will undergo some displacements, which are called strains (White, 1965) (figure 2-1). The strain has different types depending on the direction of stress applied on the body. The strain can be normal or shear. Below are a set of equations for normal and shear strains.

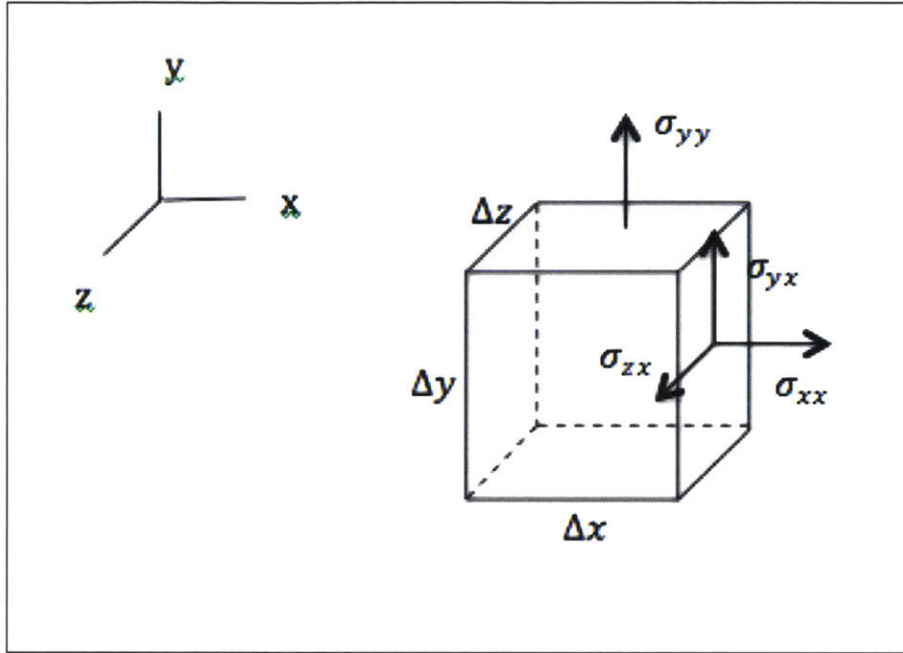


Figure 2-1: An elastic object with applied stresses

Normal in the x-, y- and z-directions:

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad (2.10)$$

Shear:

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \quad (2.11)$$

$$\epsilon_{zx} = \epsilon_{xz} = \frac{1}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right), \quad (2.12)$$

$$\epsilon_{zy} = \epsilon_{yz} = \frac{1}{2} \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right) \quad (2.13)$$

A linear relationship between stress and strain can be stated as (White, 1965):

$$\sigma = C\epsilon \quad (2.14)$$

where C is a (6×6) matrix called the Compliance. For an isotropic object, C contains 2 unique constants of proportionality known as Lamé parameters λ and μ . Below are

the equations that relate stress to strain with Lamé parameters in each direction:

$$\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz} \quad (2.15)$$

$$\sigma_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz} \quad (2.16)$$

$$\sigma_{zz} = \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{zz} \quad (2.17)$$

$$\sigma_{xy} = \mu\varepsilon_{xy} \quad (2.18)$$

$$\sigma_{xz} = \mu\varepsilon_{xz} \quad (2.19)$$

$$\sigma_{yz} = \mu\varepsilon_{yz} \quad (2.20)$$

When the stresses on the body are unbalanced, the body will no longer be in static equilibrium. This satisfies Newton's 2nd Law of motion which states that the total unbalanced forces acting on a body is equal to the product of the body's mass and its acceleration:

$$F = ma \quad (2.21)$$

The following equation is the sum of the total surface forces applied in the x-direction:

$$\sigma_{xx} = \frac{F_x}{A} \quad \rightarrow \quad F_x = \sigma_{xx}A \quad (2.22)$$

$$\begin{aligned} & (\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}\Delta x - \sigma_{xx})\Delta y\Delta z + (\sigma_{xy} + \frac{\partial\sigma_{xy}}{\partial x}\Delta y - \sigma_{xy})\Delta x\Delta z + \\ & (\sigma_{xz} + \frac{\partial\sigma_{xz}}{\partial z}\Delta z - \sigma_{xz})\Delta x\Delta y = (\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z})\Delta x\Delta y\Delta z \end{aligned} \quad (2.23)$$

$$F_x = (\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z})\Delta x\Delta y\Delta z \quad (2.24)$$

$$F_y = (\frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z})\Delta x\Delta y\Delta z \quad (2.25)$$

$$F_z = (\frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z})\Delta x\Delta y\Delta z \quad (2.26)$$

Because $m = \rho V$ where V is the volume, then $F = \rho V a$ and by using the relation

between stress, strain and displacement, assuming that there is no displacement in the y- and z-directions and that u_x is independent of u_y and u_z , the equation becomes:

$$(\lambda + 2\mu) \frac{\partial^2 u_x}{\partial x^2} = \rho \frac{\partial^2 u_x}{\partial t^2} \quad (2.27)$$

This equation is in the form of a wave equation where its velocity is:

$$V = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (2.28)$$

Lame' parameters are related to other elastic moduli such as the Bulk Modulus by: $K = \lambda + \frac{2}{3}\mu$. The velocity becomes (Sheriff and Geldart, 1995):

$$V = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}} \quad (2.29)$$

The above equation is the compressional wave velocity for 3D body with $M = K + \frac{4}{3}\mu$.

2.4 Effective Media Theories

There are many *effective media theories* that have been established during the history of studying rocks; This section reviews some of those theories.

2.4.1 Voigt and Reuss Bounds

The Voigt and Reuss bounds predict the effective elastic moduli of a material that has a mixture of phases such as rocks. The expressions provide upper and lower bounds of the elastic moduli of the material knowing the volume fraction and the elastic modulus of each phase (Mavko et al., 2003).

The Voigt upper bound is also known as the iso-strain average because it assumes that each phase experience different stresses but has the same strain. It is expressed as:

$$M_v = \sum_{i=1}^n f_i M_i \quad (2.30)$$

where M_v is Voigt effective modulus, M_i and f_i are the elastic modulus and volume fraction of each phase, respectively, and n is number of phases in the mixture.

The Reuss lower bound assumes that the layers undergo the same perpendicular stress. Hence, it is known as the iso-stress average. It is expressed as:

$$\frac{1}{M_r} = \sum_{i=1}^n \frac{f_i}{M_i} \quad (2.31)$$

2.4.2 Hashin-Shtrikman Bounds

The Hashin-Shtrikman bounds were derived in 1963 for a material with two phases (Hashin and Shtrikman, 1963). The bounds give a range of the effective elastic moduli without any information about the geometry of the medium and they are narrower than the Voigt and Reuss bounds (Mavko et al., 2003). The bounds are given by:

For the Bulk modulus:

$$K^{HS\pm} = K_1 + \frac{f_2}{(K_2 - K_1)^{-1} + f_1(K_1 + \frac{4}{3}\mu_1)^{-1}} \quad (2.32)$$

For the Shear modulus:

$$\mu^{HS\pm} = \mu_1 + \frac{f_2}{(\mu_2 - \mu_1)^{-1} + \frac{2f_1(K_1 + 2\mu_1)}{5\mu_1(K_1 + \frac{4}{3}\mu_1)}} \quad (2.33)$$

where K_1 and K_2 are the bulk moduli of each phase, μ_1 and μ_2 are the shear moduli of each phase, and f_1 and f_2 are the volume fraction of each phase.

The upper bound is computed by assuming that the stiffer phase is number 1 and the softer phase is number 2 and vice versa to calculate the lower bound.

2.4.3 Hill Average Moduli Estimate

The Hill average modulus is just the arithmetic average of the Voigt and Reuss bounds. It is expressed as:

$$M = \frac{M_v + M_r}{2} \quad (2.34)$$

where M_v and M_r are the Voigt and Reuss average described earlier in section 2.4.1, respectively (Mavko et al., 2003).

2.5 Wyllie's Equation (Time Average Equation)

In 1956 and 1958, M. R. J. Wyllie proposed a model to calculate the velocity through a porous medium with a saturating fluid in its pore space (Wyllie et al., 1956, 1958). His time average model relates the total time it takes the wave to travel across the medium to the sum of the time it takes to travel through each phase. The total time is:

$$t = t_f + t_m \quad (2.35)$$

where t_f , and t_m are the times it takes the wave to travel through the fluid and the mineral, respectively. The equation can be converted to velocities by:

$$\frac{1}{V} = \frac{\phi}{V_f} + \frac{(1 - \phi)}{V_m} \quad (2.36)$$

where V is the velocity through the medium, V_f is the velocity through the fluid filling the pore space, and V_m is the velocity through the mineral. ϕ , is the porosity, which is the ratio of pore space volume to total volume of the medium, $\phi = \frac{v_p}{v_T}$ where v_p is the pore volume and v_T is the total volume.

Chapter 3

Observations

3.1 Compressional Wave Velocity through Gaseous Elements

The fact that the Newton-Laplace equation was first derived for the sound wave travelling through air motivated me to look into the compressional wave velocity through different gases. Table 3.1 (de Podesta, 2002) has the velocities and densities of those gases, which are plotted in linear scale in figure 3-1 and log scale in figure 3-2.

The data is fitted by a power function:

$$V = a\rho^b \tag{3.1}$$

where the fit parameters are $a = 400$ and $b \approx -\frac{1}{2}$. This fit suggests that the velocity is proportional to the inverse of the square root of density. In other words, the experimental results of the compressional wave velocity through different gases are in agreement with the Newton-Laplace equation.

Table 3.1: The measured compressional wave velocities and densities for gases

Gas	Velocity (m/s)	Density (kg/m^3)
Hydrogen, H_2	1286	0.08995
Helium, He	972	0.1786
Neon, Ne	434	0.9003
Fluorine, F_2	332	1.14
Nitrogen, N_2	337	1.25
Oxygen, O_2	332	1.428
Argon, Ar	308	1.782
Bromine, Br_2	149	3.12
Chlorine, Cl_2	219	3.164
Krypton, Kr	213	3.739
Iodine, I_2	138	4.953
Xenon, Xe	170	5.858

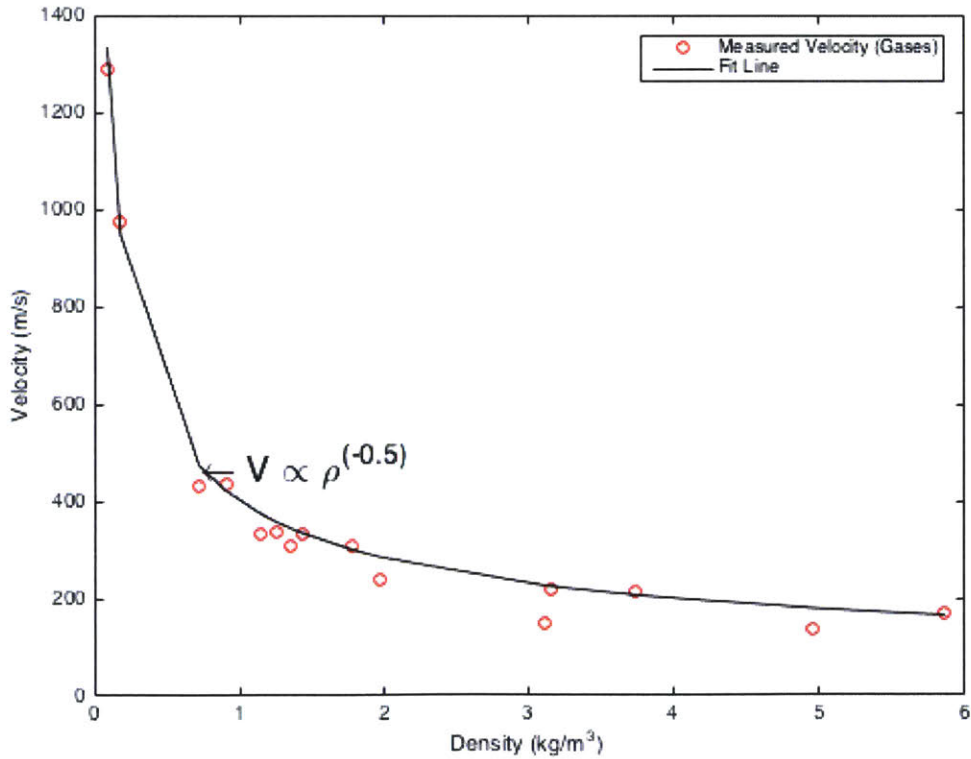


Figure 3-1: Measured velocity versus density for gaseous elements

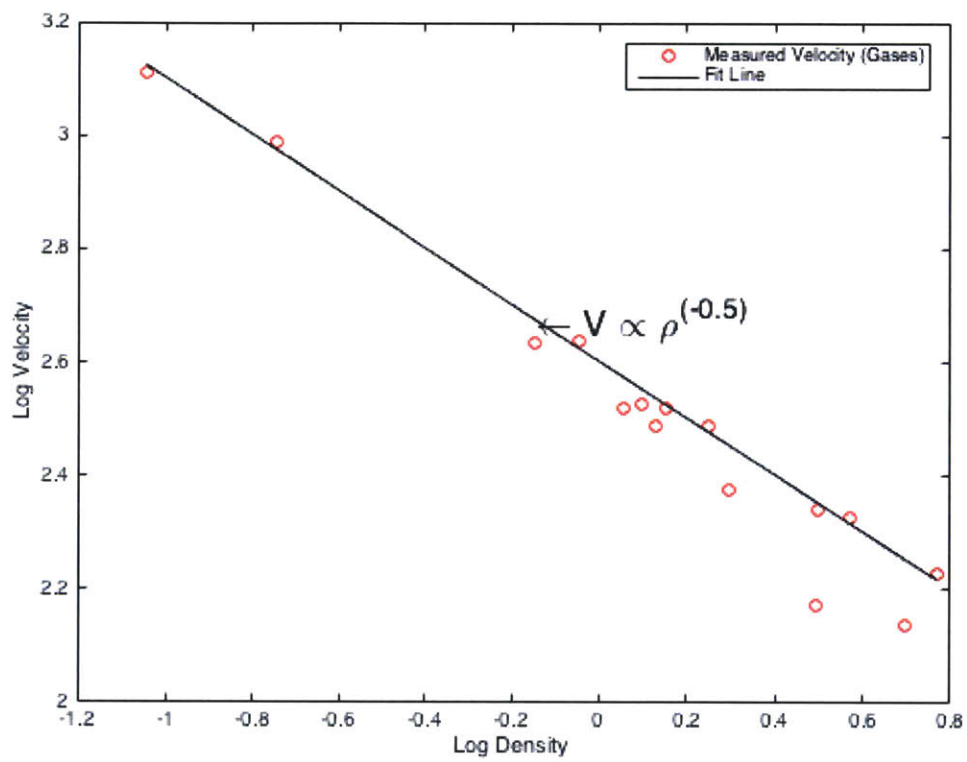


Figure 3-2: Log of measured velocity versus log of density for gaseous elements

Table 3.2: The measured compressional wave velocities and densities for non-porous solids

Solid	Velocity (m/s)	Density (kg/m^3)
Magnesium, Mg	5823	1738
Beryllium, Be	12890	1846
Aluminum, Al	6374	2698
Titanium, Ti	6130	4508
Vanadium, V	6023	6090
Zirconium, Zr	4650	6507
Zinc, Zn	4187	7135
Chromium, Cr	6608	7194
Tin, Sn	3380	7285
Manganese, Mn	4600	7473
Iron, Fe	5957	7873
Niobium, Nb	5068	8578
Cadmium, Cd	2780	8647
Nickel, Ni	5700	8907
Copper, Cu	4759	8933
Molybdenum, Mo	6475	10222
Silver, Ag	3704	10500
Lead, Pb	2160	11343
Tantalum, Ta	4159	16670
Uranium, U	3370	19050
Tungsten, W	5221	19254
Gold, Au	3240	19281
Platinum, Pt	3260	21450

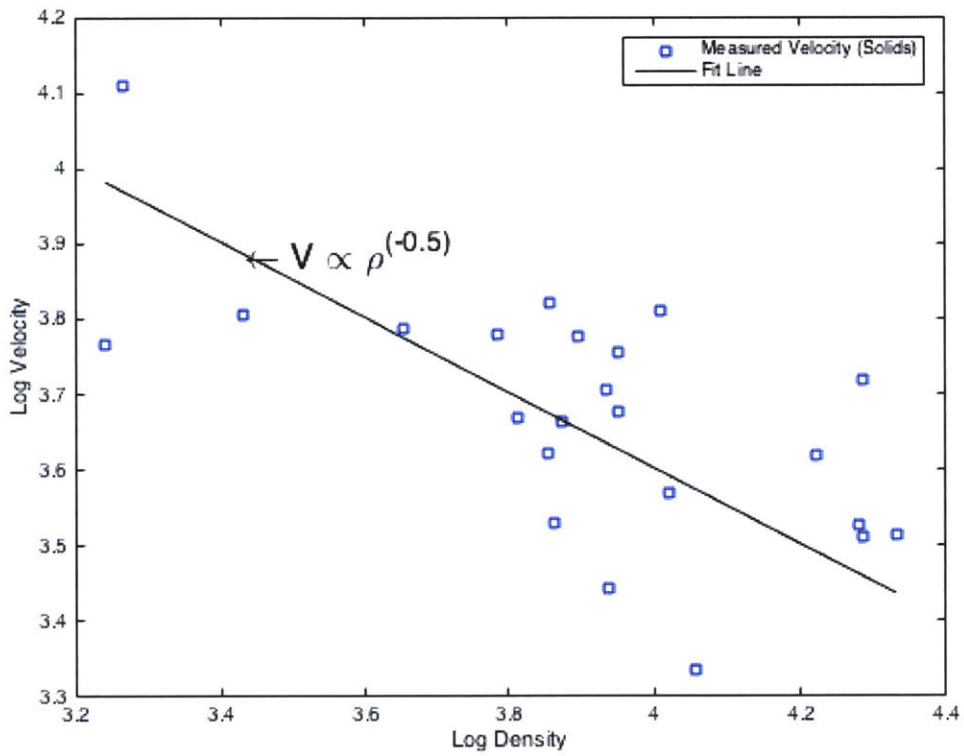


Figure 3-4: Log of measured velocity versus log of density for non-porous solid elements

Table 3.3: The average measured compressional wave velocities and average densities for rocks

Rock Type	Velocity (m/s)	Density (kg/m^3)
Dry Sand	800	1600
Wet Sand	1750	2000
Saturated Shales and Clays	1800	2200
Saturated shale and sand sections	1850	2250
Porous and saturated sandstones	2750	2250
Limestone	4750	2550
Chalk	2450	2050
Salt	5000	2200
Anhydrite	4750	2950
Dolomite	5000	2700
Granite	5250	2600
Basalt	5500	2900
Gneiss	4800	2600

3.3 Compressional Wave Velocity through Rocks

The compressional wave velocity through different rocks, shown in table 3.3 (Mavko, 2016), is plotted in linear scale in figure 3-5 and the log of velocity versus the log of density is plotted in figure 3-6. The values provided are the average for each lithology as each one has a range velocities and densities. As seen in the figure, the velocity generally increases with increasing density. The data are fitted by a power line in the form of equation 3.1, where $b \approx 2.3$.

In addition, figure 3-7 is Gardner's log-log plot that shows the average lines of the compressional wave velocity versus density for different rocks such as: Sandstone, Shale, Limestone, Dolomite, Salt and Anhydrite (Gardner et al., 1974). As seen, the compressional wave velocity is directly related to density. Gardner's average is:

$$V = 108.28\rho^4 \quad (3.2)$$

where V is in m/s and ρ is in g/cm^3 . In general, the figures for rocks are characterized by an increase in velocity with an increase in density. This opposes the expectations from the theoretical equation defined by Newton and Laplace, where the velocity is expected to decrease with increasing density.

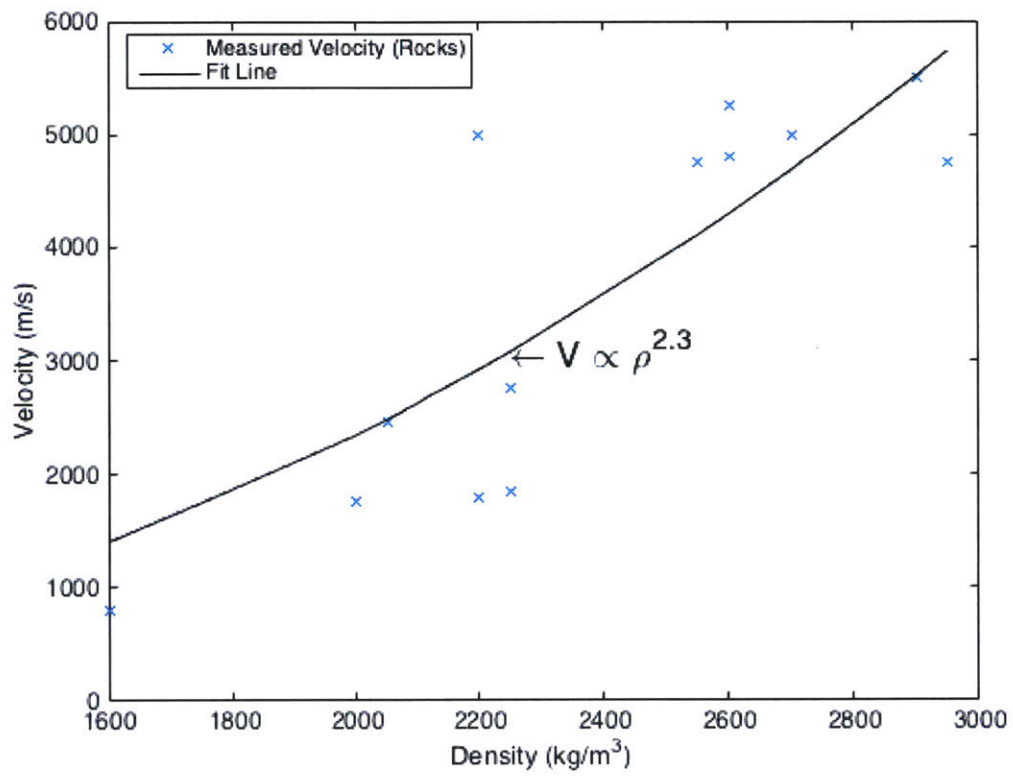


Figure 3-5: Measured velocity versus density for rocks

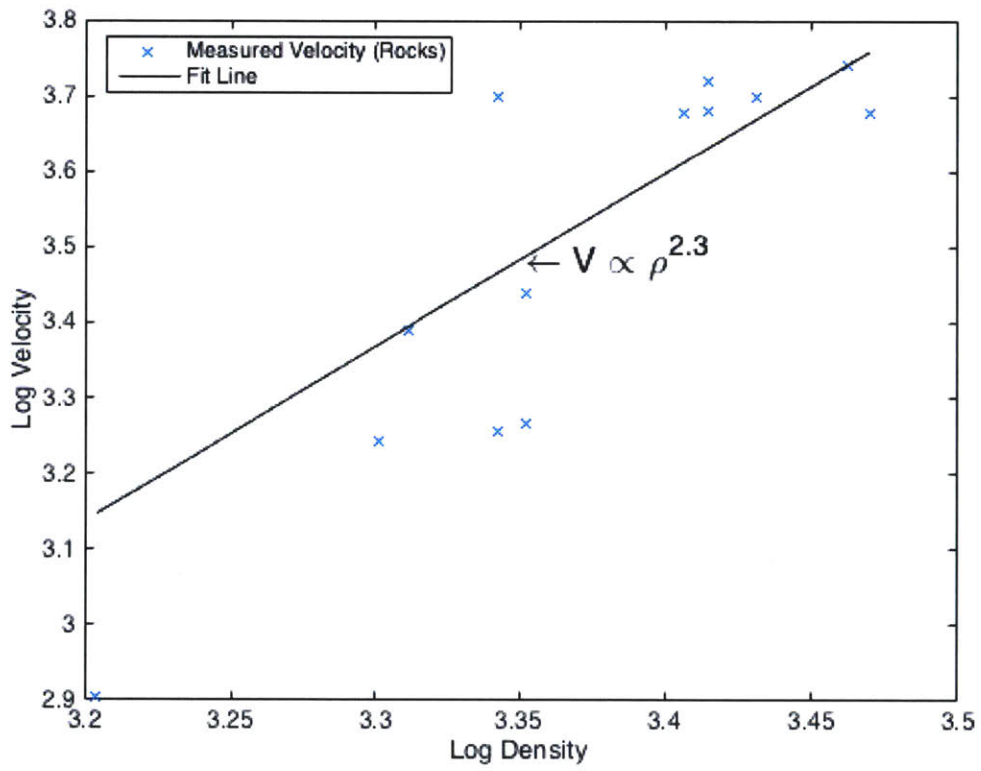


Figure 3-6: Log of measured velocity versus log of density for rocks

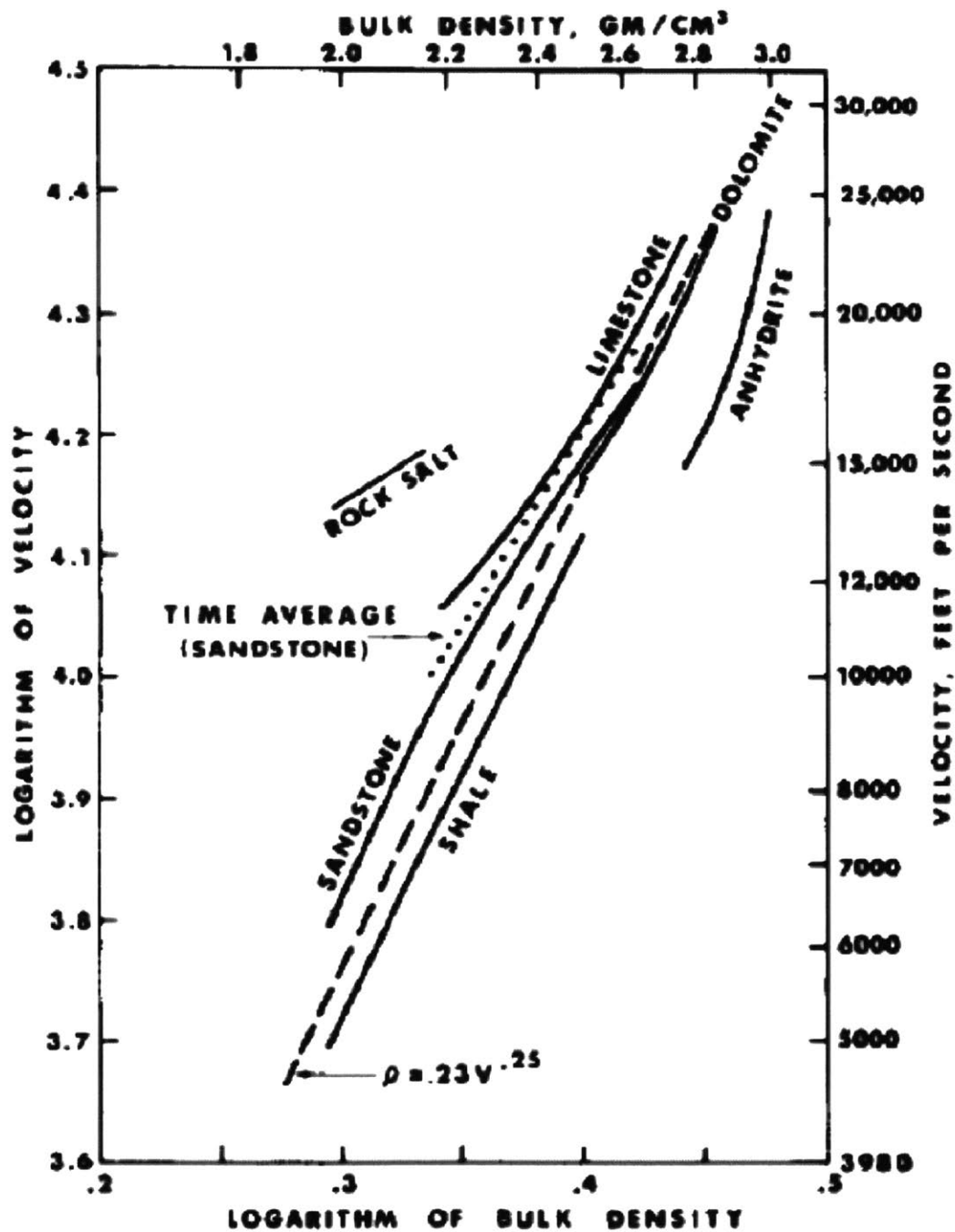


Figure 3-7: Gardner's log-log plot of velocity versus density through rocks (After (Gardner et al., 1974))

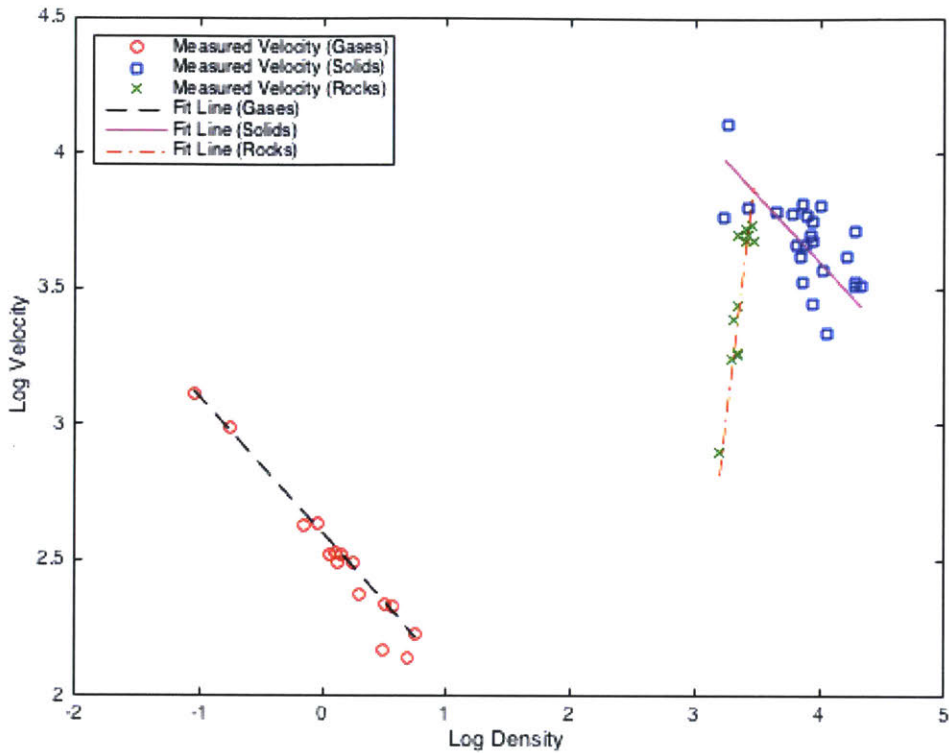


Figure 3-8: Merged log velocity versus log density for gases, solids and rocks

3.4 Conclusion

The confusion behind the relationship between velocity and density when considering different mediums can be explained by the following:

- The wave velocity through single-phase materials such as gases and non-porous solid elements conform to the Newton-Laplace theoretical expression as it is.
- The velocity through two-phase materials, such as rocks, has a different relationship to what is expected from the Newton-Laplace theory.

Figure 3-8 is a plot of log velocity versus log density for the three mediums: gases, solids and rocks. It is very clear that the single phase mediums (gases and solids) have negative slopes whereas the two-phase mediums (rocks) have positive slope. The following chapter looks at how to tackle this problem and provides a solution.

Chapter 4

Methodology

4.1 Derivation of new expression for the elastic modulus, M

The fact that the time average equation is in Gardner's plot (figure 3-7), which has a positive slope very similar to the slope of Gardner's average line, was the starting point for the derivation. The first step is to check the high power dependence of velocity on density for Wyllie's equation. This is done by calculating the velocity using equation 2.36 with the parameters in table 4.1. Figures 4-1 and 4-2 show the velocity calculated using Wyllie's equation versus density and porosity. Although rocks generally have a range of porosity from around 5% to around 35%, the velocity is calculated for porosity from 0% to 100% to see the behaviour of velocity at the two extrem ends of porosity and density.

In order to get an approximate relationship between velocity and density, matlab's curve fitting application was used to get the best-fit parameters. First, the fit was in the form of power function $V = a\rho^b$. The fit parameters are: $a = 0.006$ and $b = 1.7$ with normalized *rmse* between the calculated velocity and the fit line $rmse = 0.14$. Another fit was done using the power function $V = a\rho^b + c$. The fit parameters are: $a = 3 \times 10^{-11}$, $b = 4.1$ and $c = 1469$ with normalized *rmse* between the calculated velocity and the fit line $rmse = 0.03$, which is a better fit compared to the first form.

Table 4.1: The parameters used to calculate the velocity using Wyllie's equation

Parameter	Value
ϕ	from 0% to 100%
V_m	6000 m/s
ρ_m	2650 kg/m ³
V_f	1500 m/s
ρ_f	1000 kg/m ³

In addition, it also confirms that Wyllie's equation depends on approximately the 4th power of density, which is similar to Gardner's average.

As the velocity calculated using Wyllie's equation produces a high power dependence of velocity on density, an elastic modulus from Wyllie's equation is derived. It is basically done by setting the velocity in Wyllie's equation equal to the velocity in the Newton-Laplace form. The following steps demonstrate the derivation of the new expression of the elastic modulus, starting with Wyllie's equation:

$$\frac{1}{V} = \frac{\phi}{V_f} + \frac{(1 - \phi)}{V_m} \quad (4.1)$$

Unifying the denominator:

$$\frac{1}{V} = \frac{\phi V_m + (1 - \phi) V_f}{V_f V_m} \quad (4.2)$$

Writing it as V:

$$V = \frac{V_f V_m}{\phi V_m + (1 - \phi) V_f} \quad (4.3)$$

Our interest is defining a relationship between the elastic modulus and density, so, Wyllie's equation is rewritten in terms of density using the relationship between porosity and density. The relationship between porosity and density is established by the weighted average formula to calculate the bulk density of rocks:

$$\rho = \phi \rho_f + (1 - \phi) \rho_m \quad (4.4)$$

where ρ is the bulk density, ρ_m is the density of the mineral matrix and ρ_f is the

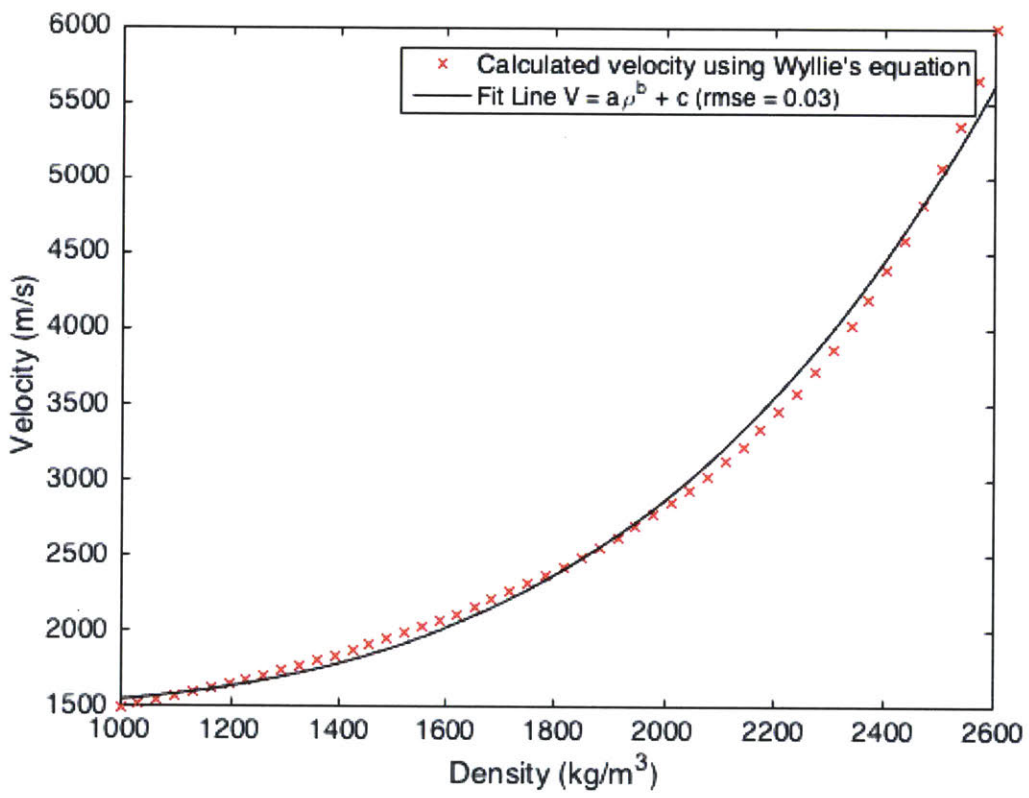


Figure 4-1: The calculated velocity using Wyllie's equation versus density along with the best-fit line

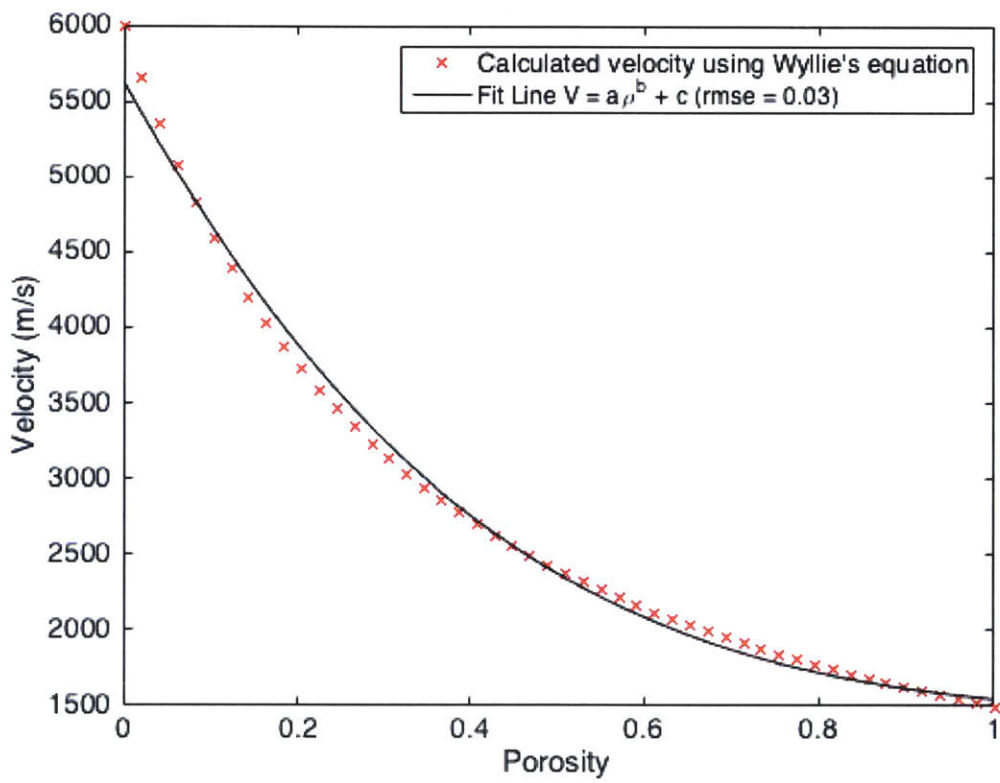


Figure 4-2: The calculated velocity using Wyllie's equation versus porosity along with the best-fit line

density of the fluid in the pore space. This makes the porosity in terms of density:

$$\phi = \frac{\rho - \rho_m}{\rho_f - \rho_m} \quad (4.5)$$

Using the above equation 4.5 yields:

$$V = \frac{V_f V_m}{\left(\frac{\rho_m - \rho}{\rho_m - \rho_f}\right) V_m + \left(1 - \left(\frac{\rho_m - \rho}{\rho_m - \rho_f}\right)\right) V_f} \quad (4.6)$$

Taking $\left(\frac{1}{\rho_m - \rho_f}\right)$ as a common factor in the denominator:

$$V = \frac{V_f V_m}{\left(\frac{1}{\rho_m - \rho_f}\right) [(\rho_m - \rho)(V_m - V_f)] + V_f} \quad (4.7)$$

Multiplying the equation by $\frac{(\rho_m - \rho_f)}{(\rho_m - \rho_f)}$ gives:

$$V = \frac{(\rho_m - \rho_f) V_f V_m}{[(\rho_m - \rho)(V_m - V_f)] + (\rho_m - \rho_f) V_f} \quad (4.8)$$

Expanding the denominator gives:

$$V = \frac{(\rho_m - \rho_f) V_f V_m}{\rho_m V_m - \rho_m V_f - \rho V_m + \rho V_f + \rho_m V_f - \rho_f V_f} \quad (4.9)$$

The $\rho_m V_f$ terms in the denominator cancel each other, the result is:

$$V = \frac{(\rho_m - \rho_f) V_f V_m}{\rho_m V_m - \rho V_m + \rho V_f - \rho_f V_f} \quad (4.10)$$

Simplifying the denominator:

$$V = \frac{(\rho_m - \rho_f) V_f V_m}{\rho(V_f - V_m) + \rho_m V_m - \rho_f V_f} \quad (4.11)$$

Setting the above equation equal to the velocity in the Newton-Laplace form:

$$\sqrt{\frac{M}{\rho}} = \frac{(\rho_m - \rho_f) V_f V_m}{\rho(V_f - V_m) + \rho_m V_m - \rho_f V_f} \quad (4.12)$$

Square both sides of the equation:

$$\frac{M}{\rho} = \left[\frac{(\rho_m - \rho_f)V_f V_m}{\rho(V_f - V_m) + \rho_m V_m - \rho_f V_f} \right]^2 \quad (4.13)$$

$$\frac{M}{\rho} = \frac{(\rho_m - \rho_f)^2 V_f^2 V_m^2}{(\rho(V_f - V_m) + \rho_m V_m - \rho_f V_f)^2} \quad (4.14)$$

Solving for the elastic modulus, M , gives the new expression in terms of bulk density:

$$M(\rho) = \frac{(\rho_m - \rho_f)^2 V_f^2 V_m^2 \rho}{(\rho(V_f - V_m) + \rho_m V_m - \rho_f V_f)^2} \quad (4.15)$$

In addition, keeping Wyllie's equation in terms of porosity gives:

$$M(\phi) = \frac{\phi \rho_f V_f^2 V_m^2 + (1 - \phi) \rho_m V_f^2 V_m^2}{(V_f + \phi(V_m - V_f))^2} \quad (4.16)$$

The above two equations 4.15 and 4.16 represent the elastic modulus of Wyllie's equation that provides the anticipated high power dependence of velocity on density.

4.2 Applying the new expression to synthetic data

The new expression is used to calculate the elastic modulus using the parameters in table 4.1. Figures 4-3 and 4-4 are plots of the calculated elastic modulus, M , using the new derived expression in equation 4-3 versus density and porosity. The new expression provides an exact relationship between the elastic modulus and density. In order to get an numerical approximate relationship between M and density, the approximated relationship between the velocity using Wyllie's equation and the Newton-Laplace equation is used to derived the expected dependence between M and density. The steps below demonstrate this:

Starting with the power function that was used to fit Wyllie's equation:

$$V = a\rho^b + c \quad (4.17)$$

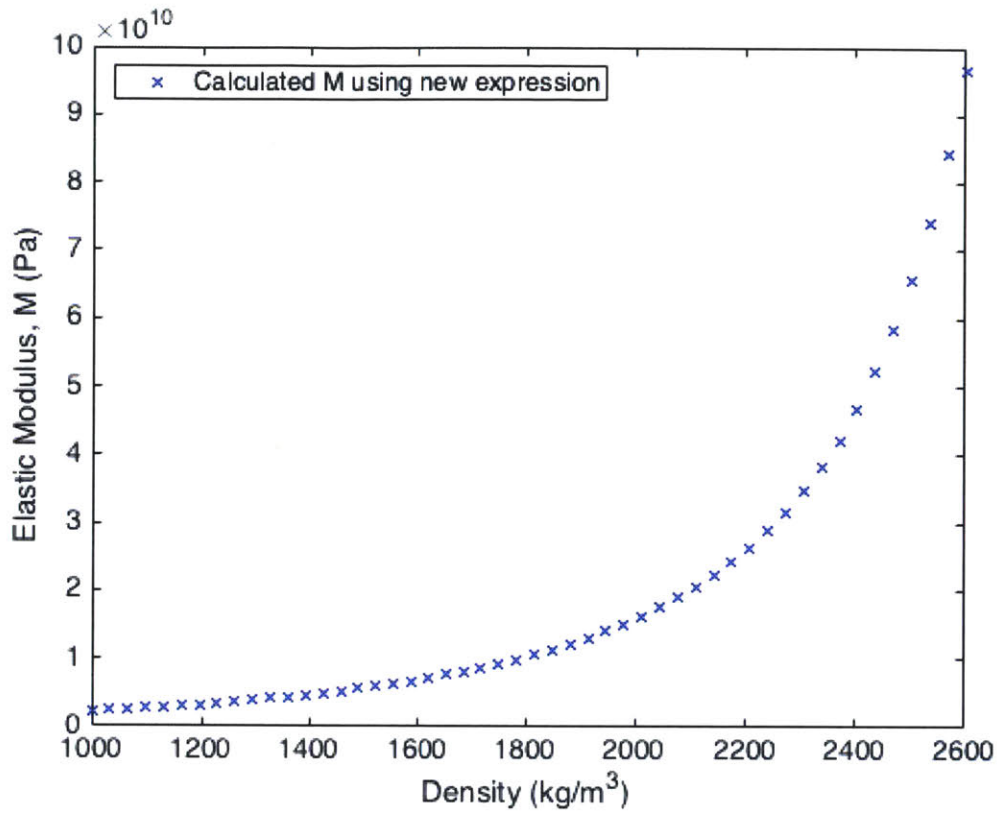


Figure 4-3: The calculated elastic modulus using the new derived expression versus density

Using the Newton-Laplace equation in order to get the elastic modulus, M, and setting the velocities equal to each other:

$$\sqrt{\frac{M}{\rho}} = a\rho^b + c \tag{4.18}$$

Solving for M by squaring both sides:

$$\frac{M}{\rho} = (a\rho^b + c)^2 \tag{4.19}$$

Solving for M:

$$M = (a\rho^b + c)^2\rho \tag{4.20}$$

The form of equation 4.20 is used to get the fit parameters, which are: $a =$

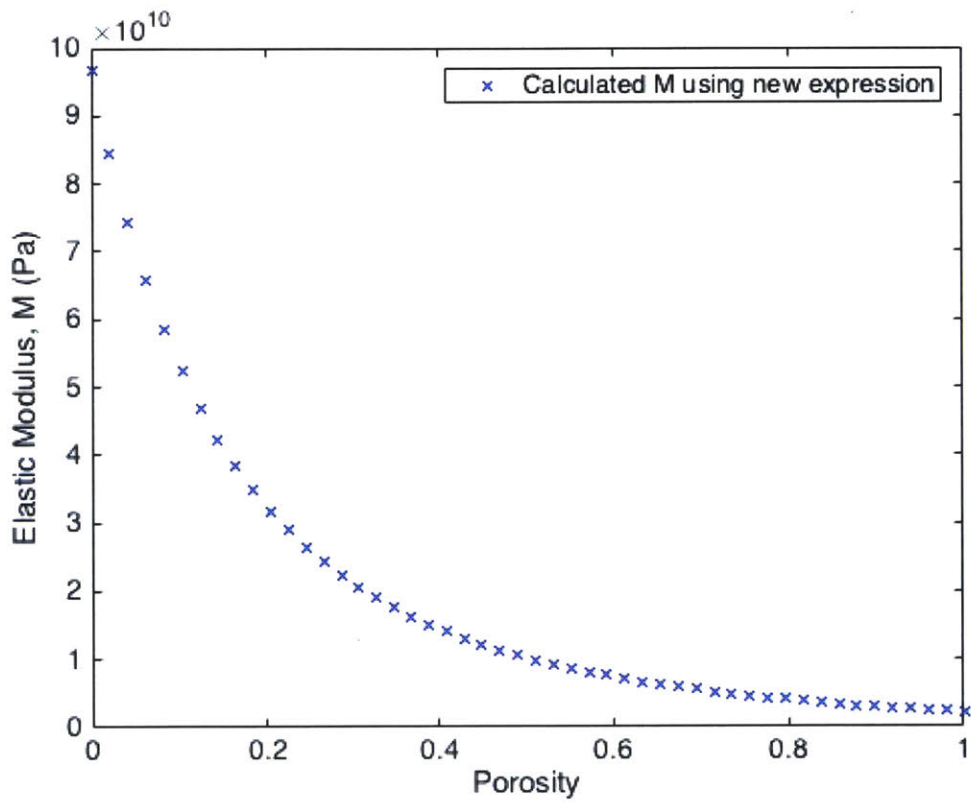


Figure 4-4: The calculated elastic modulus using the new derived expression versus porosity

1.2×10^{-10} , $b = 3.96$ and $c = 1455$ with normalized $rmse = 0.06$. The parameters are close to the parameters for Wyllie's equation. Moreover, expanding equation 4.20 and using $b = 4$ produces a relationship between the elastic modulus and density as:

$$M = A\rho^9 + B\rho^5 + C\rho, \quad (4.21)$$

where A , B , and C are constants. As seen, the elastic modulus is proportional to the power 9 of density.

The previous approximation is a little bit complicated. So, to further approximate the relationship between the elastic modulus and density in a simpler form, a power function $M = a\rho^b$, where $b = 9$, is used to get the fit parameter $a = 1.795 \times 10^{-20}$. Figure 4-5 is a plot of the calculated elastic modulus using the new expression, the fit line $M = (a\rho^b + c)^2\rho$ and the fit line $M = a\rho^b$ versus density. It is very clear that the fit line $M = (a\rho^b + c)^2\rho$ gives good approximation of the derived elastic modulus, M .

4.3 Comparing the elastic modulus using the new expression to the elastic modulus using the effective media theories and the geometric mean

This section shows the results of comparing the elastic modulus using the new expression to the the elastic modulus using the effective media theories and the geometric mean. The models were introduced in chapter 2 and they are:

- Voigt upper and Reuss lower bounds
- Hashin-Shtrikman upper and lower bounds
- Hill average estimate

In addition, the geometric mean is another way of averaging the elastic modulus

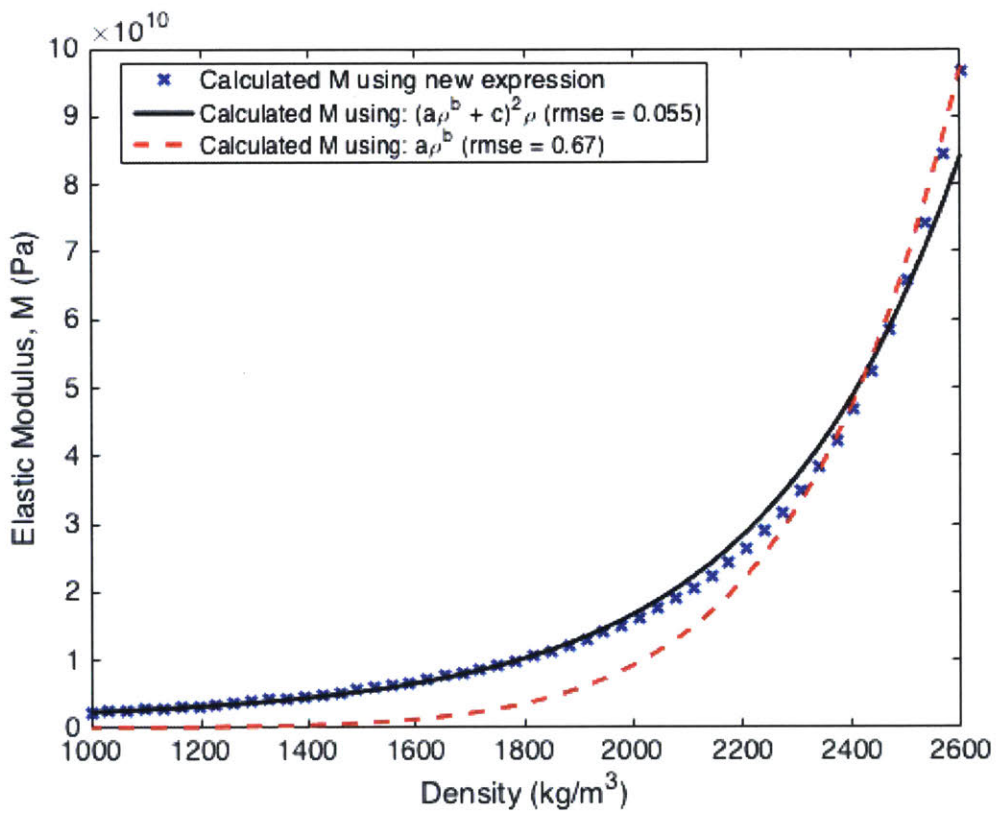


Figure 4-5: The calculated elastic modulus using the new derived expression, calculated elastic modulus using $[M = (a\rho^b + c)^2\rho, rmse = 0.05]$ and the calculated elastic modulus using $[M = a\rho^b, rmse = 0.67]$ versus density

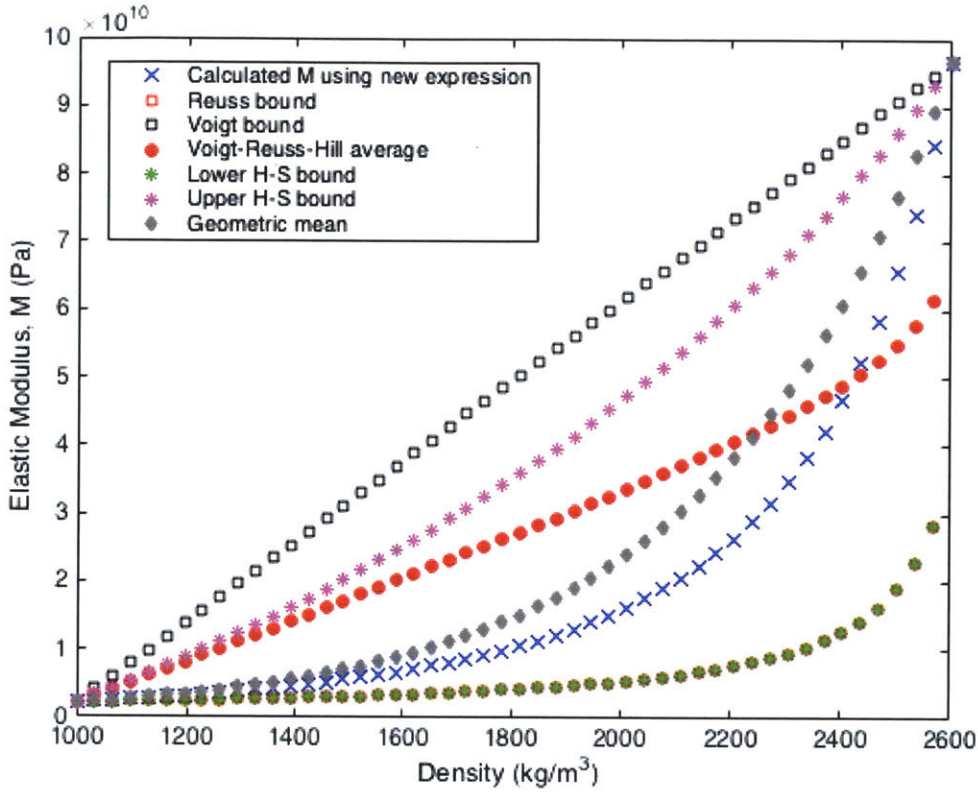


Figure 4-6: The calculated effective elastic modulus using: the new derived expression, effective media theories and the geometric mean versus density

of the rock. It is expressed as:

$$M_{gm} = M_m^{(1-\phi)} \times M_f^\phi \quad (4.22)$$

where M_{gm} is the elastic modulus of the rock, M_m is the elastic modulus of the mineral, M_f is the elastic modulus of the fluid and ϕ is the porosity.

Table 4.1 contains all parameters used in the calculations of the effective elastic modulus. The results of each effective modulus model were then used to calculate the velocity using the Newton-Laplace equation. Figure 4-6 and 4-7 are plots of the calculated elastic modulus versus density and versus porosity using the above models along with the calculated elastic modulus using the new derived expression and the geometric mean. The lower Hashin-Shtrikman and Reuss bounds are similar.

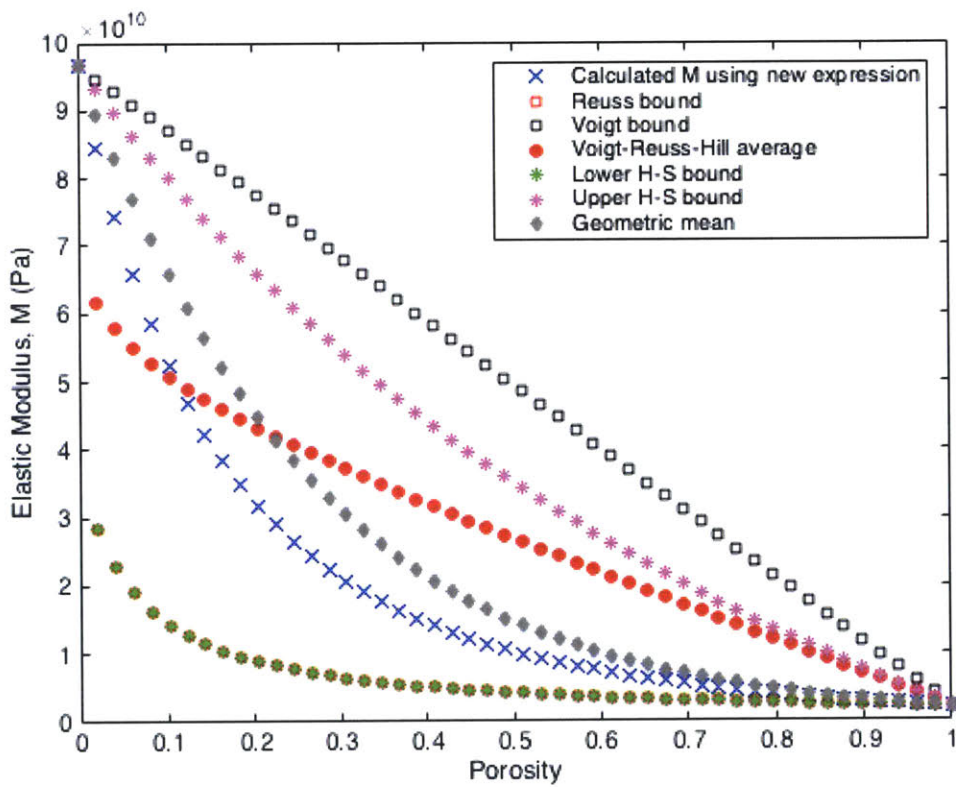


Figure 4-7: The calculated effective elastic modulus using: the new derived expression, effective media theories and the geometric mean versus porosity

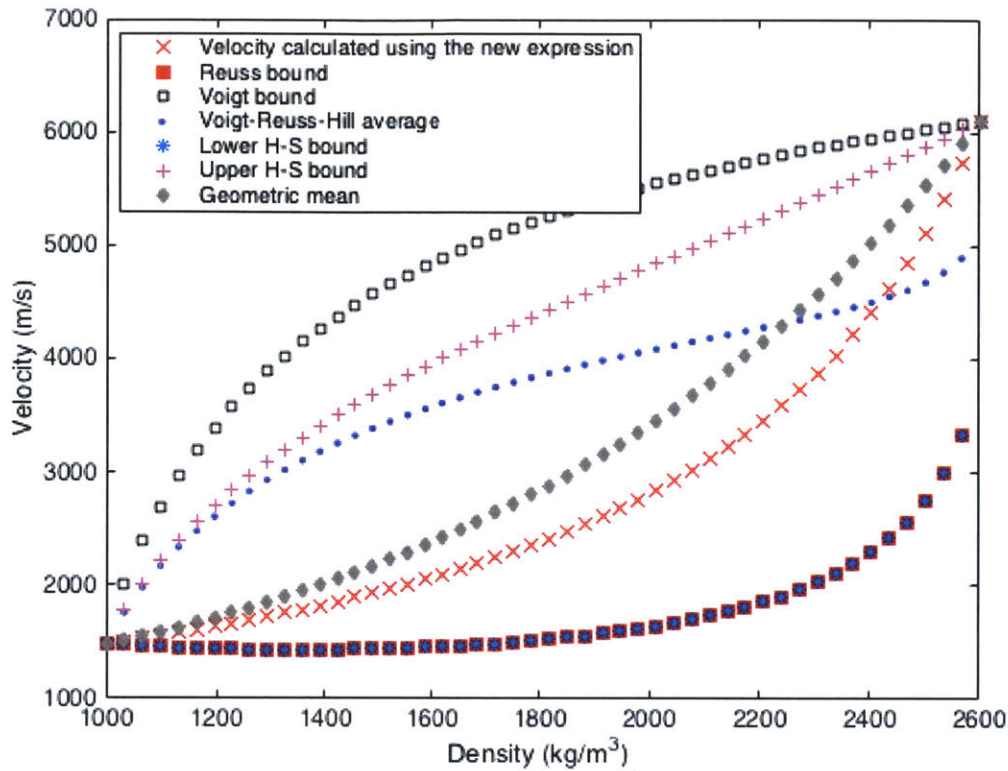


Figure 4-8: The calculated velocity using the elastic modulus calculated by: the new derived expression, effective media theories and the geometric mean versus density

Moreover, Figures 4-8 and 4-9 are plots of the calculated velocity versus density and versus porosity using the above models along with the velocity calculated using the new derived expression of M and the geometric mean.

As seen, the elastic modulus using the effective media theories do not produce the expected high power dependence of velocity on density. On the other hand, the geometric mean is very close to the new expression. The normalized *rmse* between the geometric mean and the calculated elastic modulus using the new expression is 0.32.

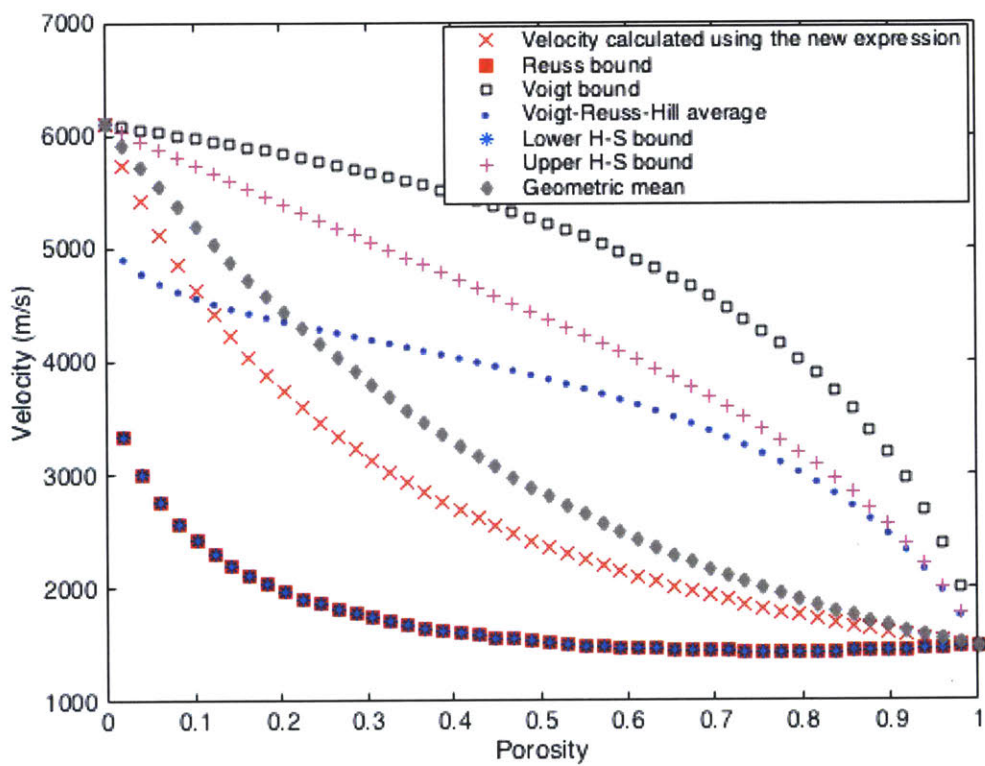


Figure 4-9: The calculated velocity using: the new derived expression, effective media theories and the geometric mean versus porosity

4.4 Generalized Gardner's equation

It was stated previously that Gardner's equation has the form $V = a\rho^b$, where $b = 4$. However, using the same form of Gardner's equation to get an approximated relationship between the velocity using Wyllie's equation and density did not produce the power 4 dependence. On the other hand, the other form with the added constant ($V = a\rho^b + c$) gave the 4 power dependence of velocity on density. This fact introduced the idea of modifying Gardner's equation. In addition, the modified or generalized form will ensure that the velocity is correct throughout the entire range of densities. As seen in figure 4-10, Gardner's velocity does not provide the correct value at the lower end of density (density of water), which is 1500 m/s . The modified form is:

$$V = a\rho^b + c \quad (4.23)$$

The new additional term c is constrained by the fact that at the density of water (1.00 g/cm^3), the velocity should be equal to the velocity of water which is 1500 m/s . This is obtained by constraining the sum of a and c to be 1500 ($a + c = 1500$). This will improve the fit to the calculated data as well as insure that the velocity is correct for the entire range of densities.

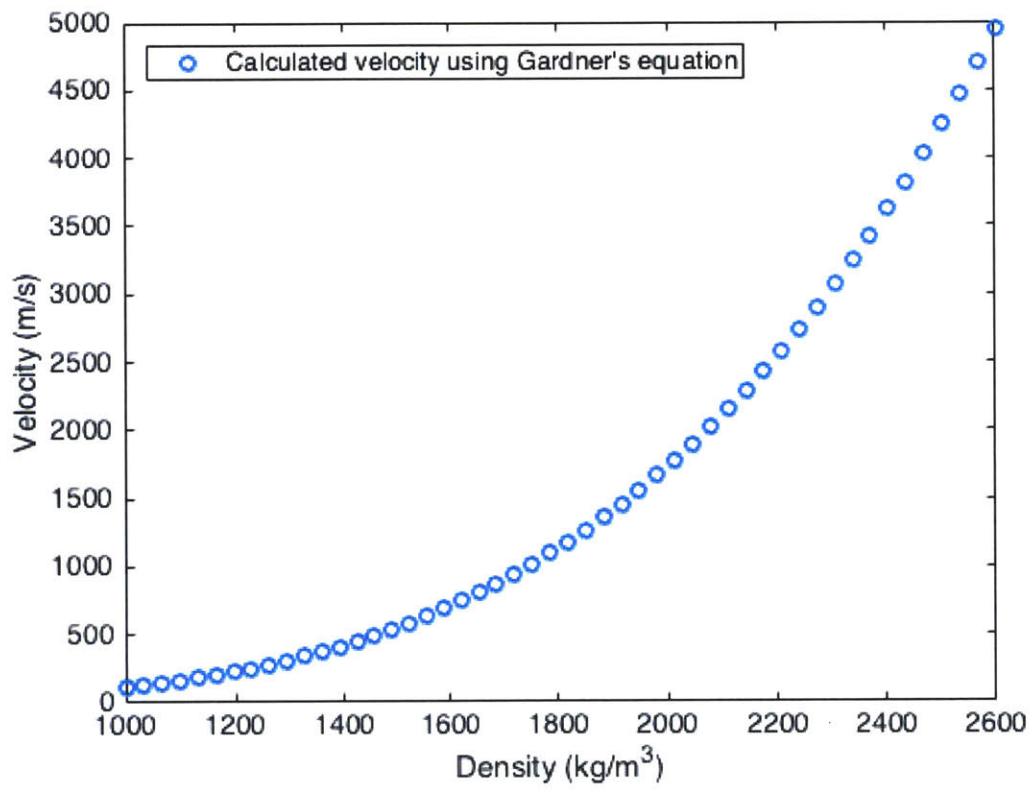


Figure 4-10: The calculated velocity using Gardner's equation versus density

Chapter 5

Applications to Real Data

5.1 Introduction

The data used in this chapter are well-log data that includes: compressional wave velocity, bulk density, porosity, and the volume of each mineral at each measurement. Three out of the four wells are mainly Carbonate rocks provided by Saudi Aramco (well No. 1, well No. 2 and well No. 3). Well No. 4 has Clastic rocks, that is taken from the problem sets in the book *Quantitative Seismic Interpretation (QSI)* (Avseth et al., 2005). The figures from 5-1 to 5-4 are plots of the velocity versus density for each well.

5.2 Applying the new expression of the elastic modulus, M , to real data

This section contains the results of applying the new expression of the elastic modulus, M , (equation 4.15) that was derived in section 4.1 to real data sets. Each well in Saudi Aramco's data was divided by the dominant mineral at each measurement. For example, if there is at least 90% Calcite and a maximum of 10% Anhydrite and Dolomite, the measurement is considered within the Calcite or Limestone interval. On the other hand, the data from the *QSI* book are already divided into 6 intervals

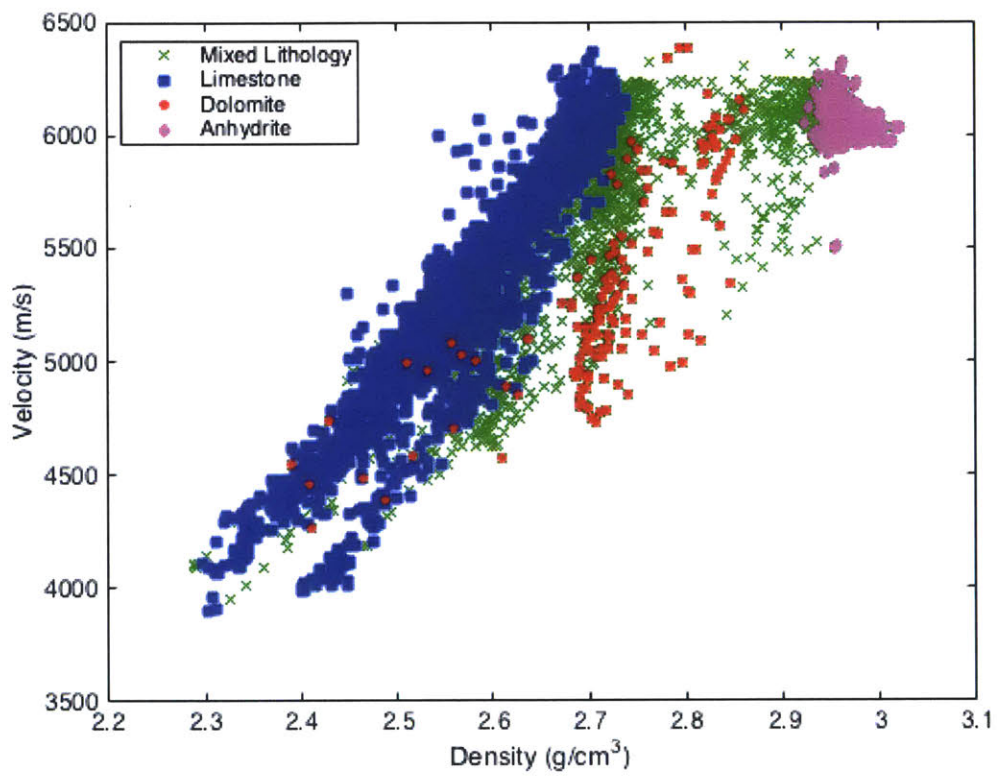


Figure 5-1: The velocity versus density for well No. 1

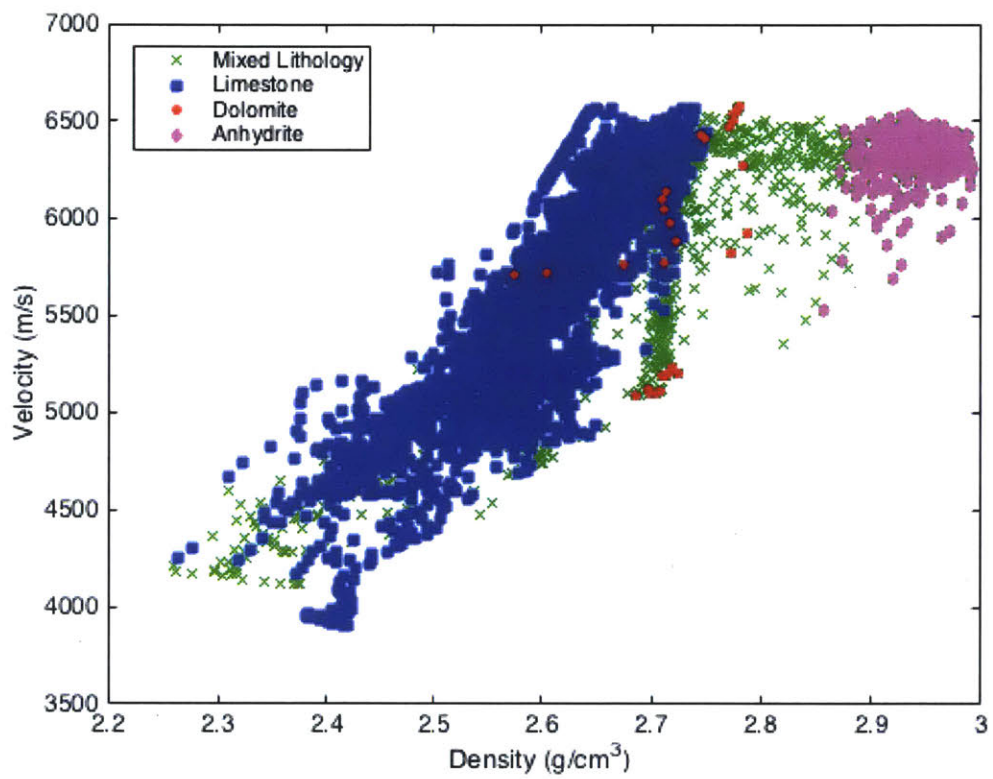


Figure 5-2: The velocity versus density for well No. 2

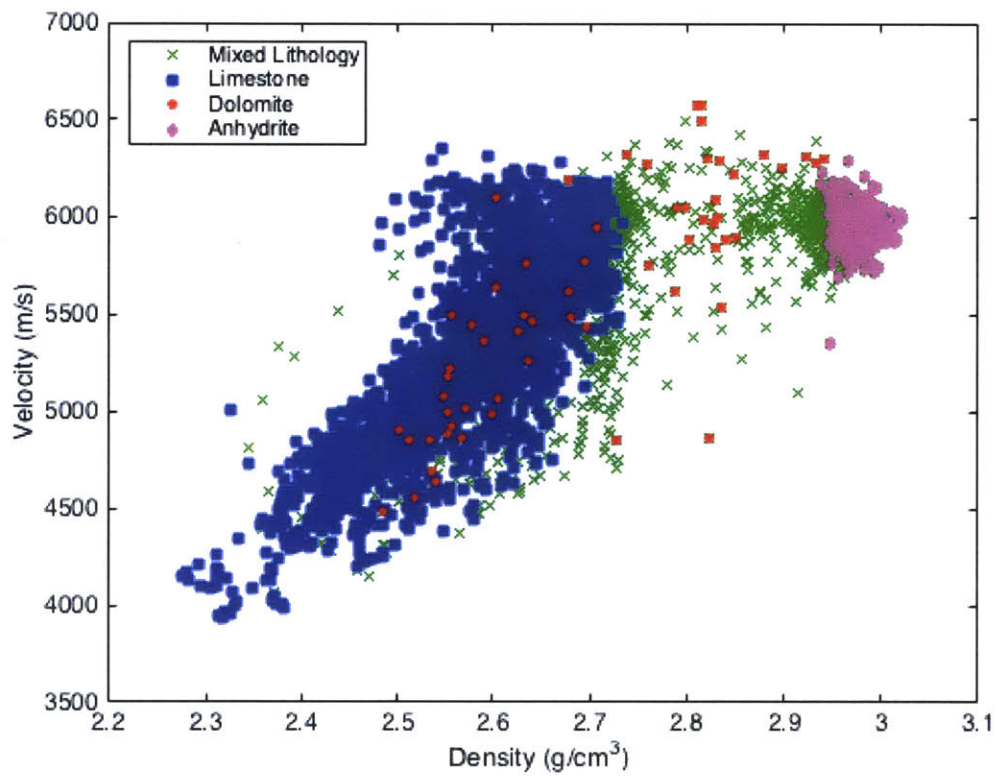


Figure 5-3: The velocity versus density for well No. 3

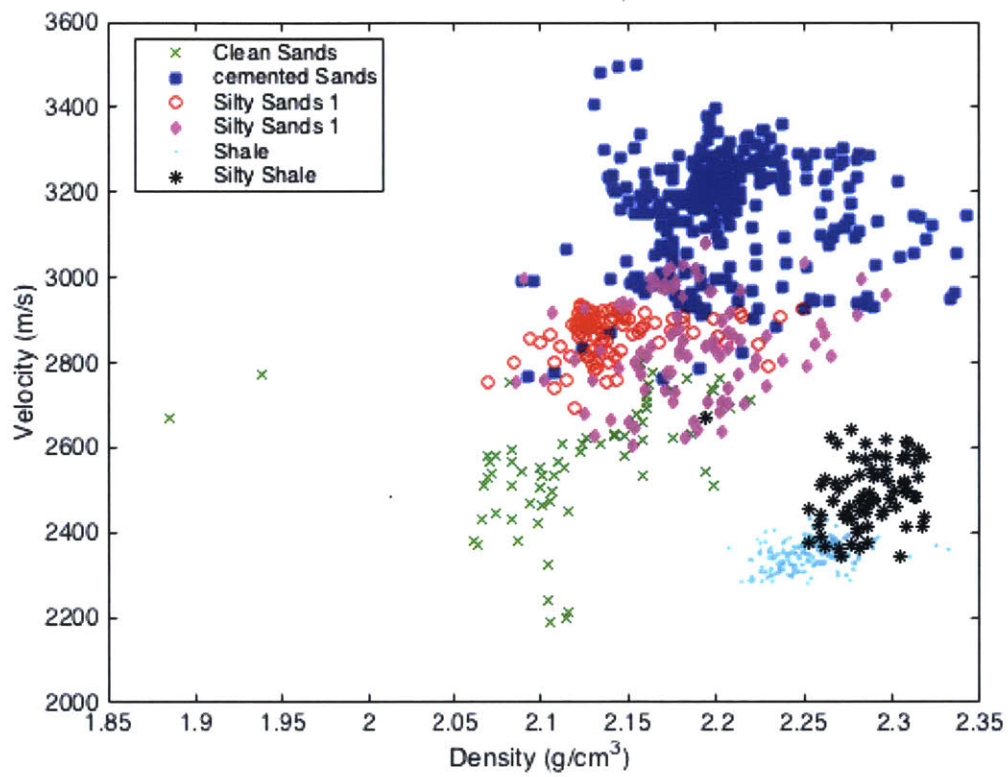


Figure 5-4: The velocity versus density for well No. 4

Table 5.1: The average of the parameters used in calculating the velocity using the new derived expression of the elastic modulus

Medium	Parameter	Value
Calcite	V_m	6000 <i>m/s</i>
	ρ_m	2.7 <i>g/cm³</i>
Dolomite	V_m	7000 <i>m/s</i>
	ρ_m	2.7 <i>g/cm³</i>
Quartz	V_m	5000 <i>m/s</i>
	ρ_m	2.65 <i>g/cm³</i>
Water	V_f	1500 <i>m/s</i>
	ρ_f	1 <i>g/cm³</i>

based on the lithology of that interval. However, only one lithology is investigated in this section (Sandstone) as the other intervals are either silty sands, cemented sands or silty shale. Introducing the cement or silt factor might affect the values used for the mineral parameters, which in turn might impact the results of the calculations.

Table 5.1 contains the average parameters used in the calculations for each lithology interval and for water. The figures from 5-5 to 5-16 represent the results of the calculated velocity using the new derived expression of the elastic modulus, M , for each interval within the four wells. In addition, table 5.2 has the calculated *rmse* values for each well.

5.3 Applying the generalized form of Gardner's equation to real data

This section contains the results of applying the generalized form of Gardner's equation (equation 4.23) to each lithology within each well. Table 5.3 has the results of the fit parameters "a" and "b", where $c = 1500 - a$. Moreover, the figures from 5-17 to 5-28 show the calculated velocity using the generalized form of Gardner's equation

Table 5.2: The calculated normalized *rmse* between the calculated velocity using the new expression of *M* and the measured velocity for each lithology interval

Well No.	Lithology	rmse
Well 1	Limestone	0.03
	Dolomite	0.05
Well 2	Limestone	0.06
Well 3	Limestone	0.07
	Dolomite	0.10
Well 4	Clean Sands	0.05

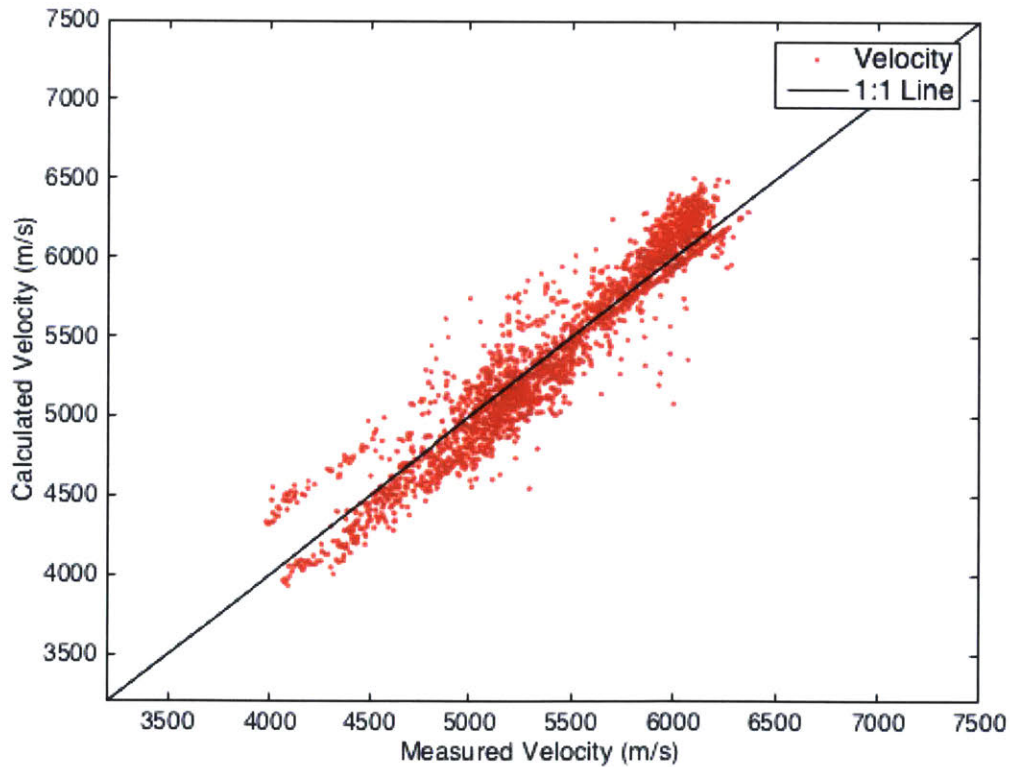


Figure 5-5: The calculated velocity using the new derived expression of the elastic modulus, *M*, versus measured velocity for Limestone within Well No. 1 (*rmse* = 0.03)

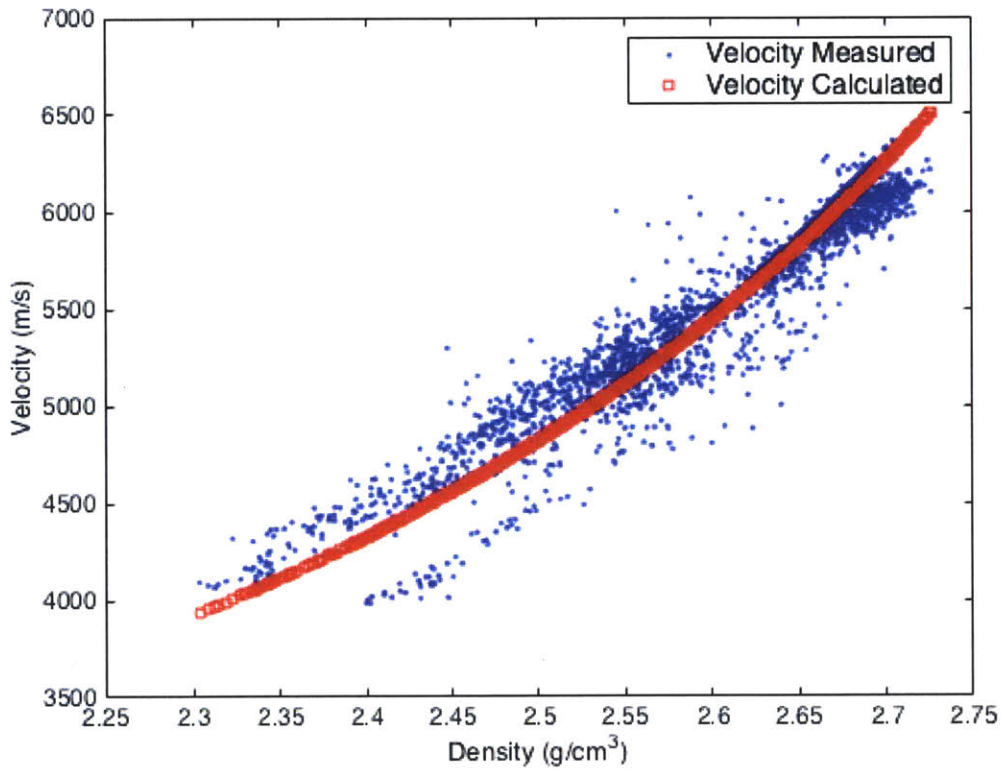


Figure 5-6: The calculated velocity using the new derived expression of the elastic modulus, M , and measured velocity versus density for Limestone within Well No. 1 ($rmse = 0.03$)

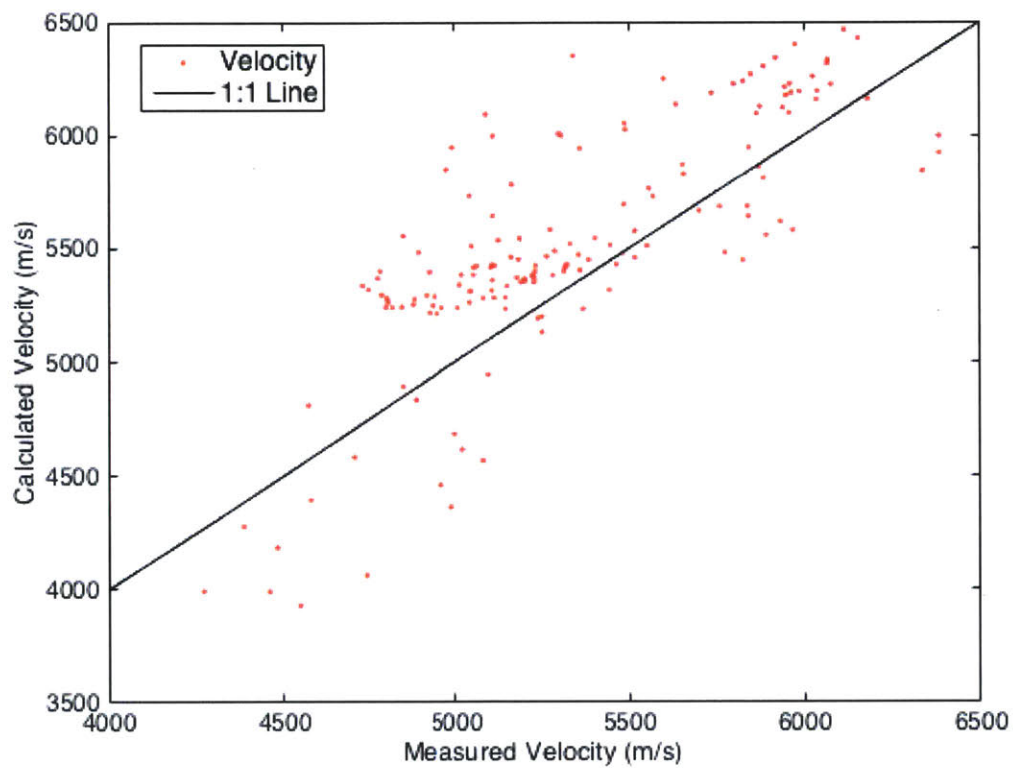


Figure 5-7: The calculated velocity using the new derived expression of the elastic modulus, M , versus measured velocity for Dolomite within Well No. 1 ($rmse = 0.05$)

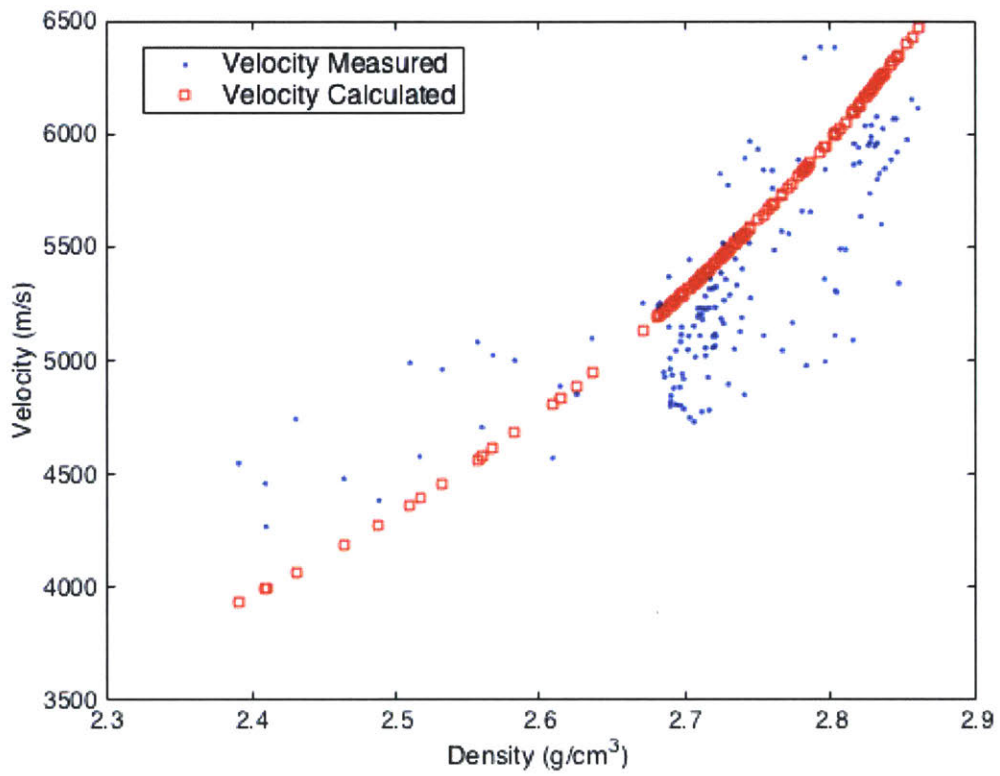


Figure 5-8: The calculated velocity using the new derived expression of the elastic modulus, M , and measured velocity versus density for Dolomite within Well No. 1 ($rmse = 0.05$)

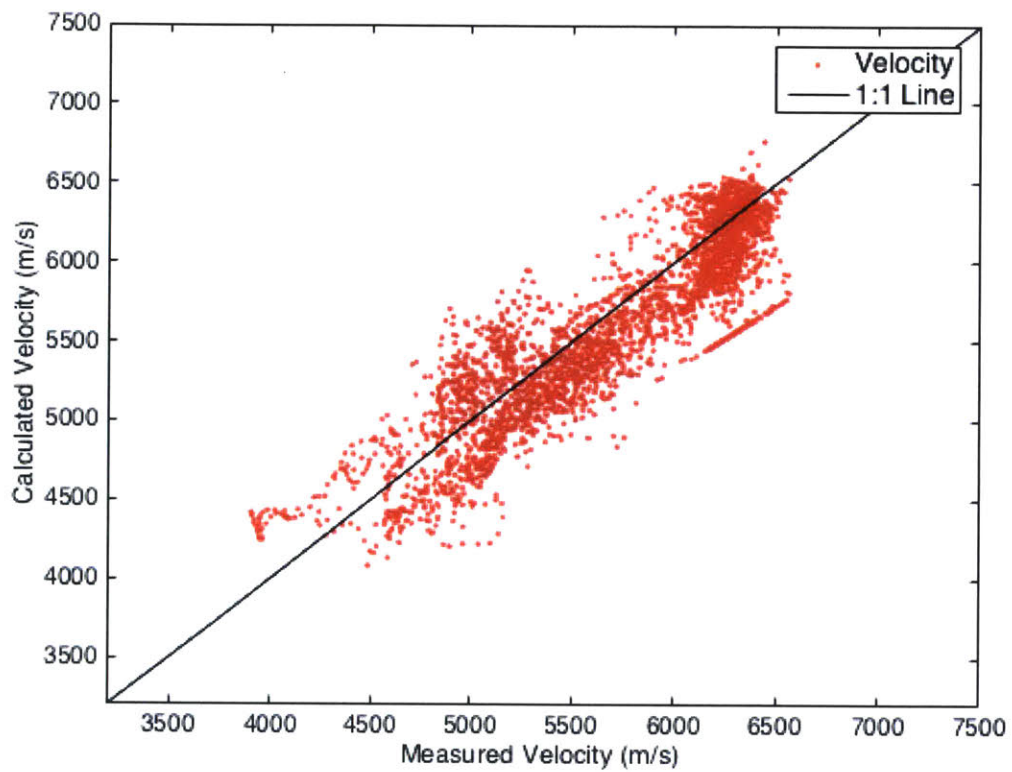


Figure 5-9: The calculated velocity using the new derived expression of the elastic modulus, M , versus measured velocity for Limestone within Well No. 2 ($rmse = 0.06$)

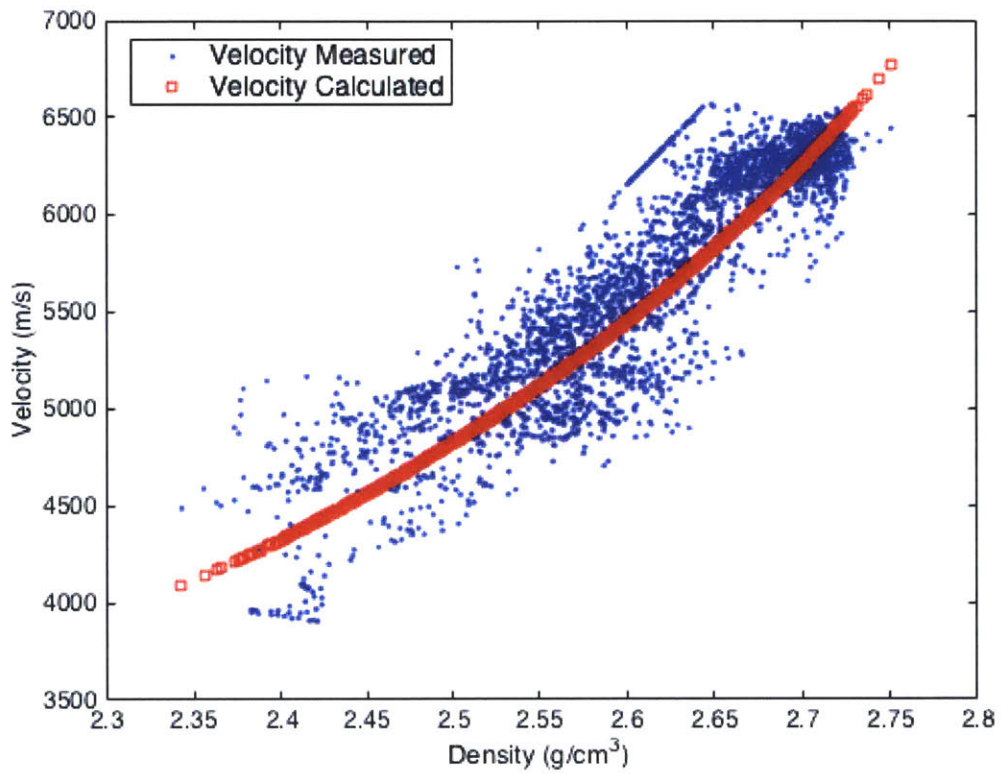


Figure 5-10: The calculated velocity using the new derived expression of the elastic modulus, M , and measured velocity versus density for Limestone within Well No. 2 ($rmse = 0.06$)

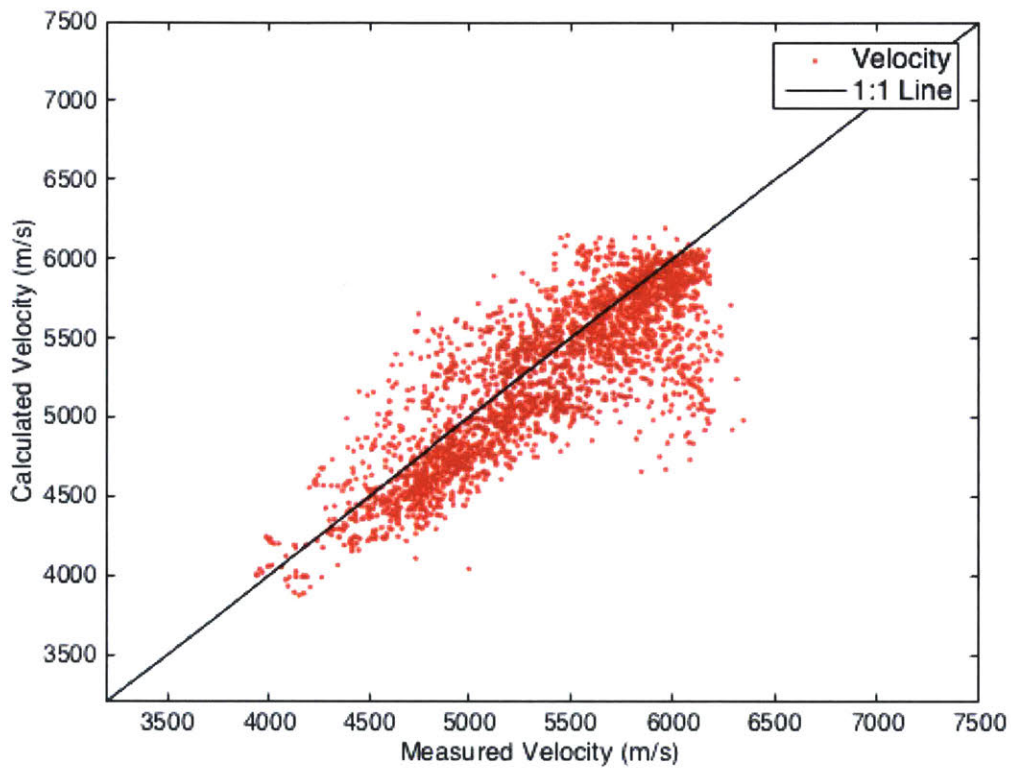


Figure 5-11: The calculated velocity using the new derived expression of the elastic modulus, M , versus measured velocity for Limestone within Well No. 3 ($rmse = 0.07$)

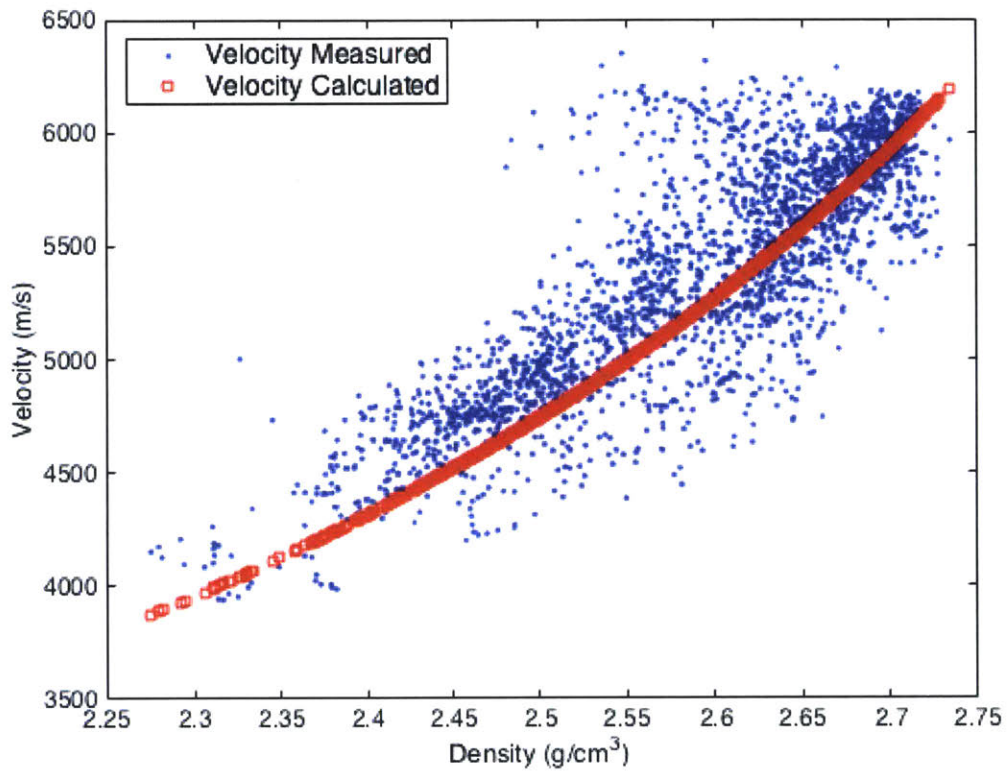


Figure 5-12: The calculated velocity using the new derived expression of the elastic modulus, M , and measured velocity versus density for Limestone within Well No. 3 ($rmse = 0.07$)

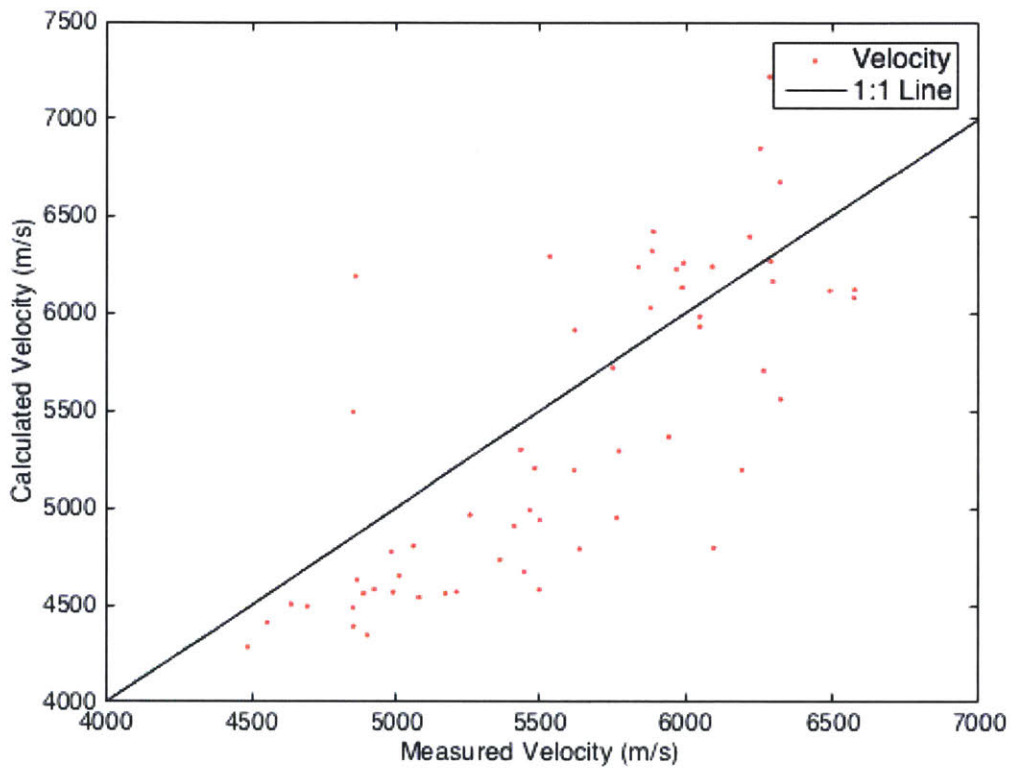


Figure 5-13: The calculated velocity using the new derived expression of the elastic modulus, M , versus measured velocity for Dolomite within Well No. 3 ($rmse = 0.10$)

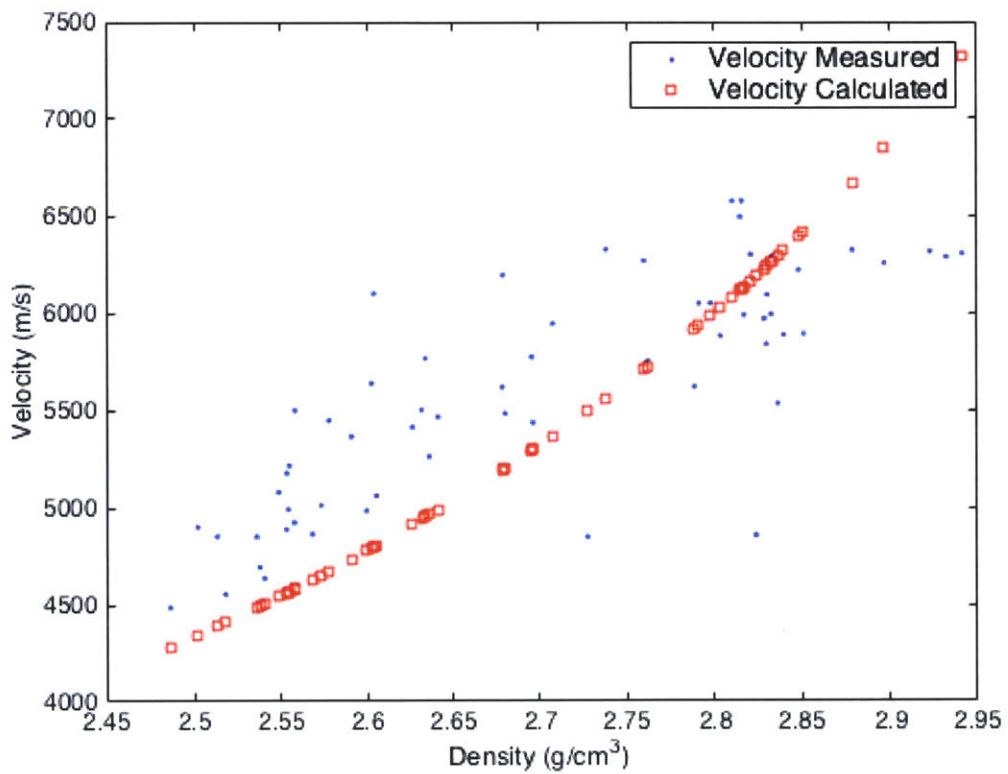


Figure 5-14: The calculated velocity using the new derived expression of the elastic modulus, M , and measured velocity versus density for Dolomite within Well No. 3 ($rmse = 0.10$)

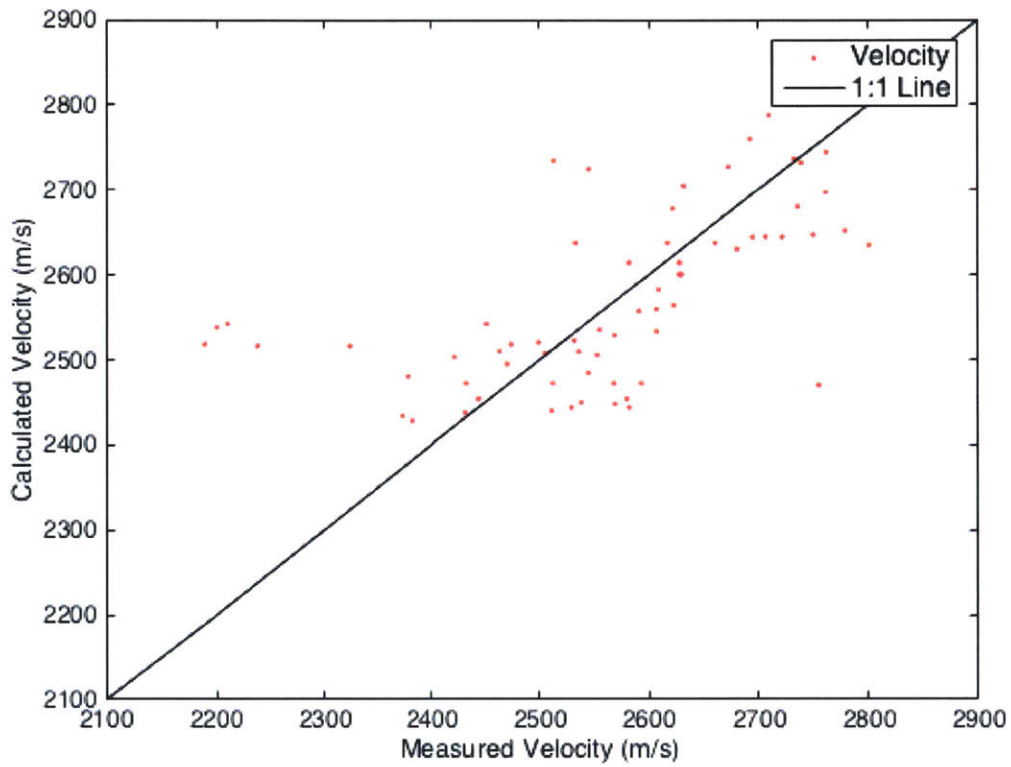


Figure 5-15: The calculated velocity using the new derived expression of the elastic modulus, M , versus measured velocity for Clean Sands within Well No. 4 ($rmse = 0.05$)

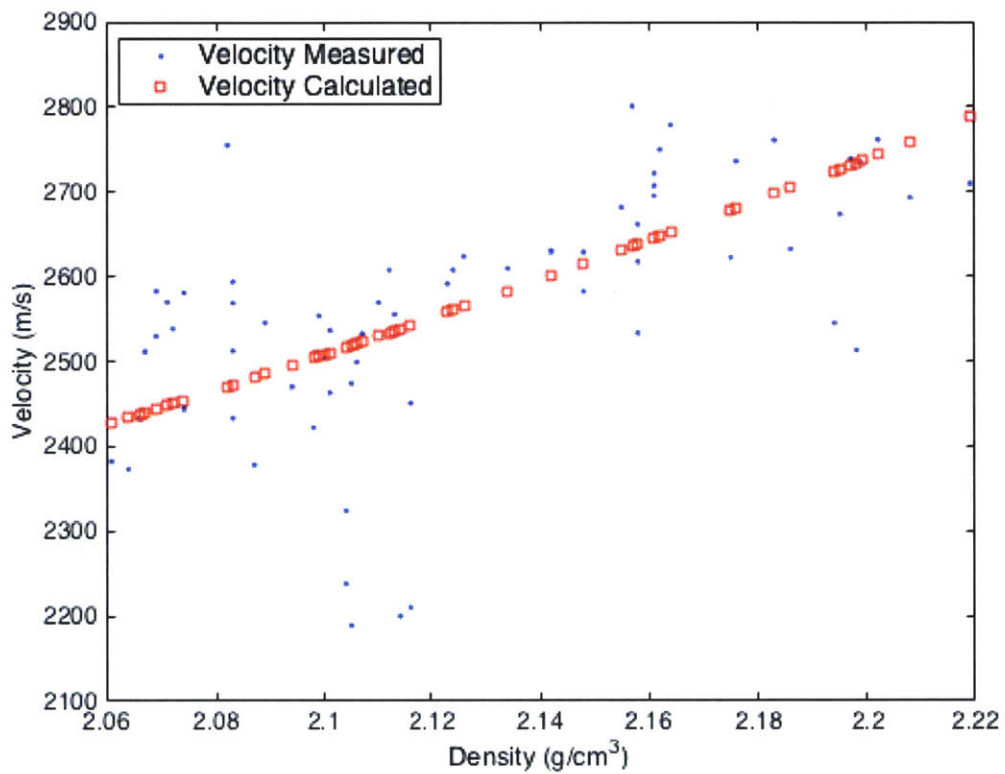


Figure 5-16: The calculated velocity using the new derived expression of the elastic modulus, M , and measured velocity versus density for Clean Sands within Well No. 4 ($rmse = 0.05$)

Table 5.3: The fit parameters "a" and "b" and the normalized *rmse* for each lithology

Well No.	Lithology	a	b	c = 1500 - a	rmse
Well 1	Limestone	102	3.0	1398	0.03
	Dolomite	336	2.5	1164	0.05
Well 2	Limestone	78	4.2	1422	0.05
Well 3	Limestone	228	3.0	1272	0.05
	Dolomite	272	2.8	1228	0.06
Well 4	Clean Sands	59	4.1	1451	0.05

versus measured velocity and the calculated and measured velocities versus density for each lithology interval. It is very clear that the results of calculating the velocity in the generalized form of Gardner's equation fit the measured data with the highest *rmse* around 6%.

In addition, to find the average value for the fit parameter "b" for all different lithologies, a weighted average is calculated: $b_{average} = \frac{\sum(n_i \times b_i)}{N}$ where n_i is the number of points within an interval, b_i is the value of the fit parameter b for that interval, and N is the total number of points. The result is $b_{average} = 3.7$, which is close to the power found by Gardner and the fit of Wyllie's equation.

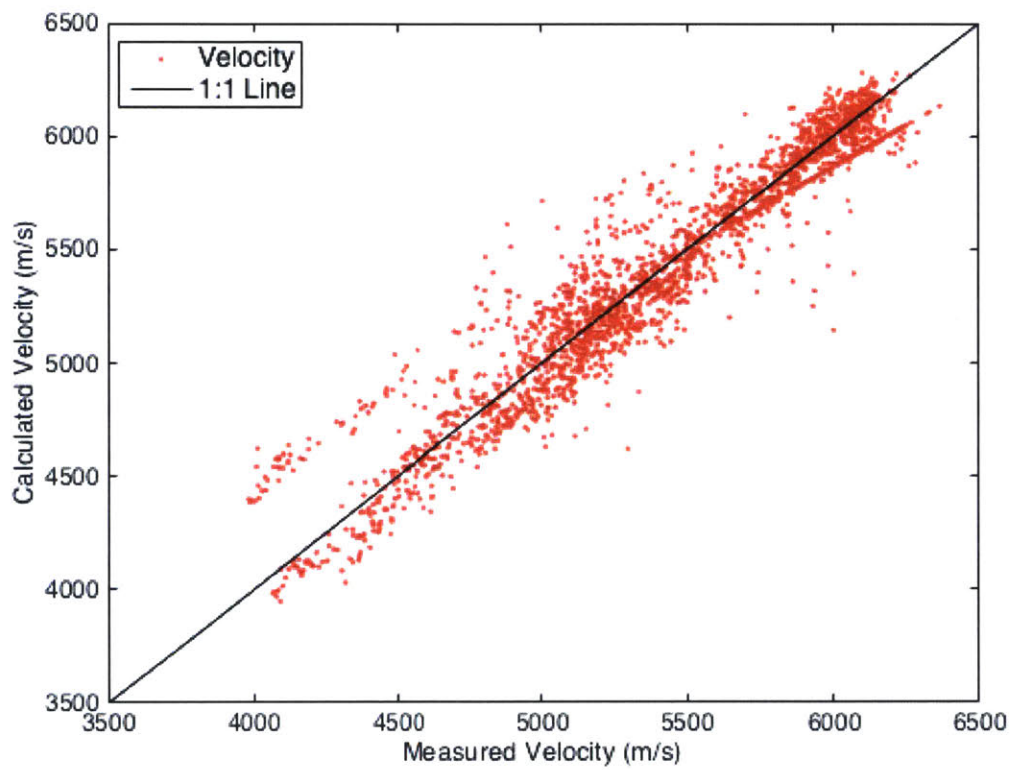


Figure 5-17: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Limestone within Well No. 1 ($rmse = 0.03$)

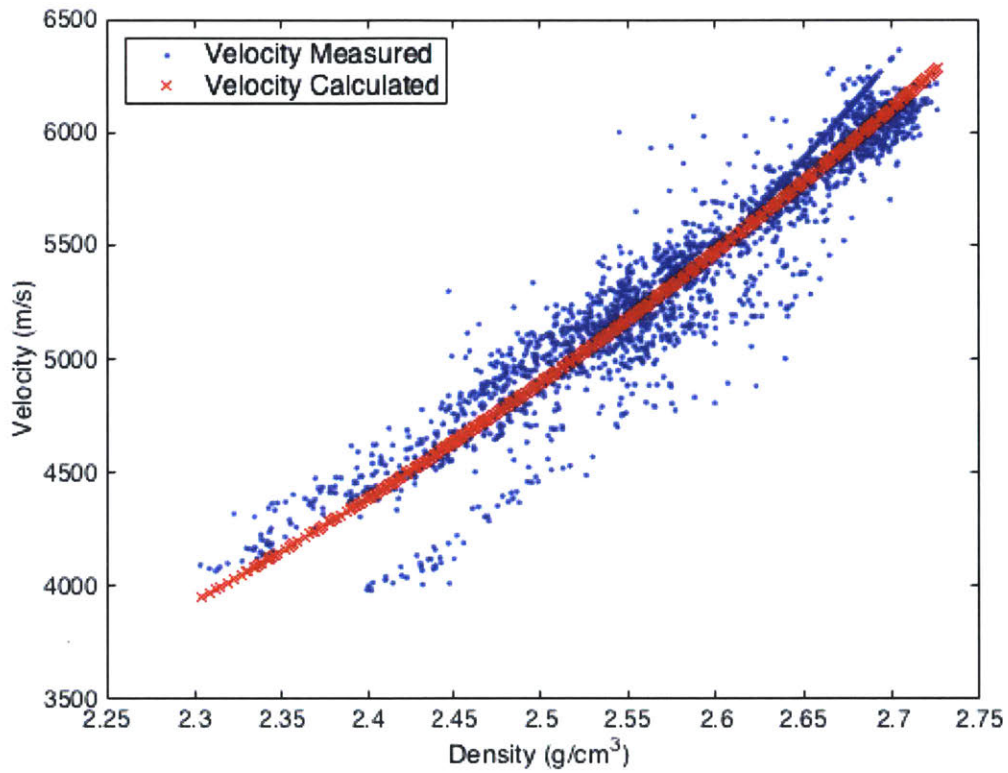


Figure 5-18: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Limestone within Well No. 1 ($rmse = 0.03$)

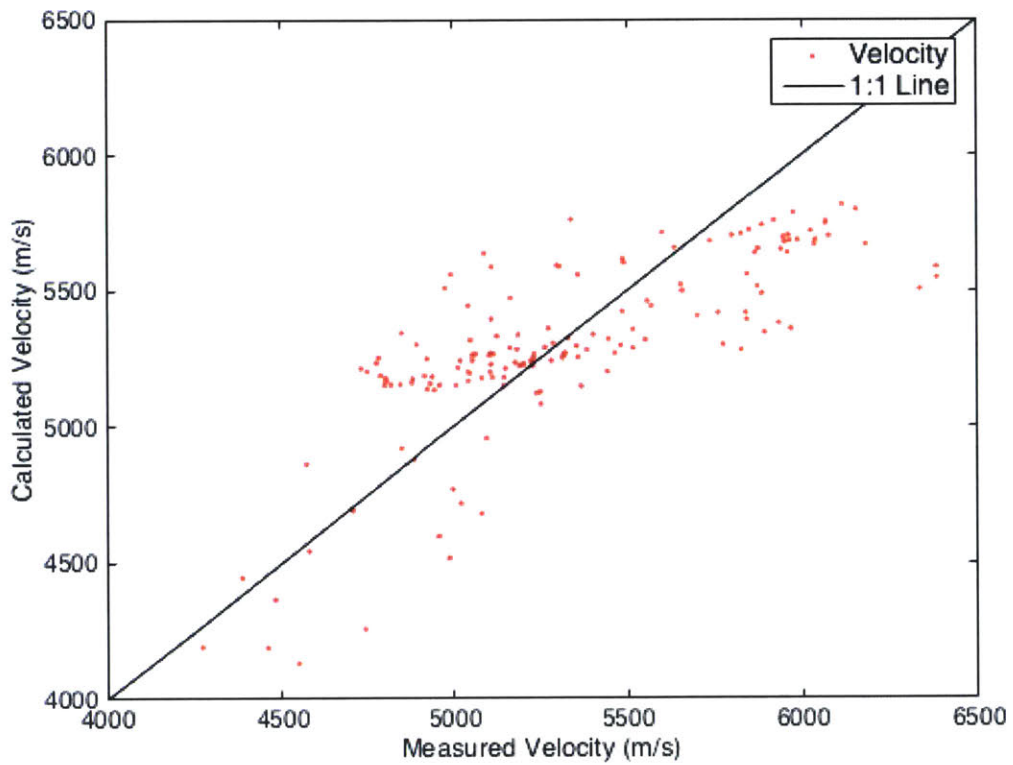


Figure 5-19: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Dolomite within Well No. 1 ($rmse = 0.05$)

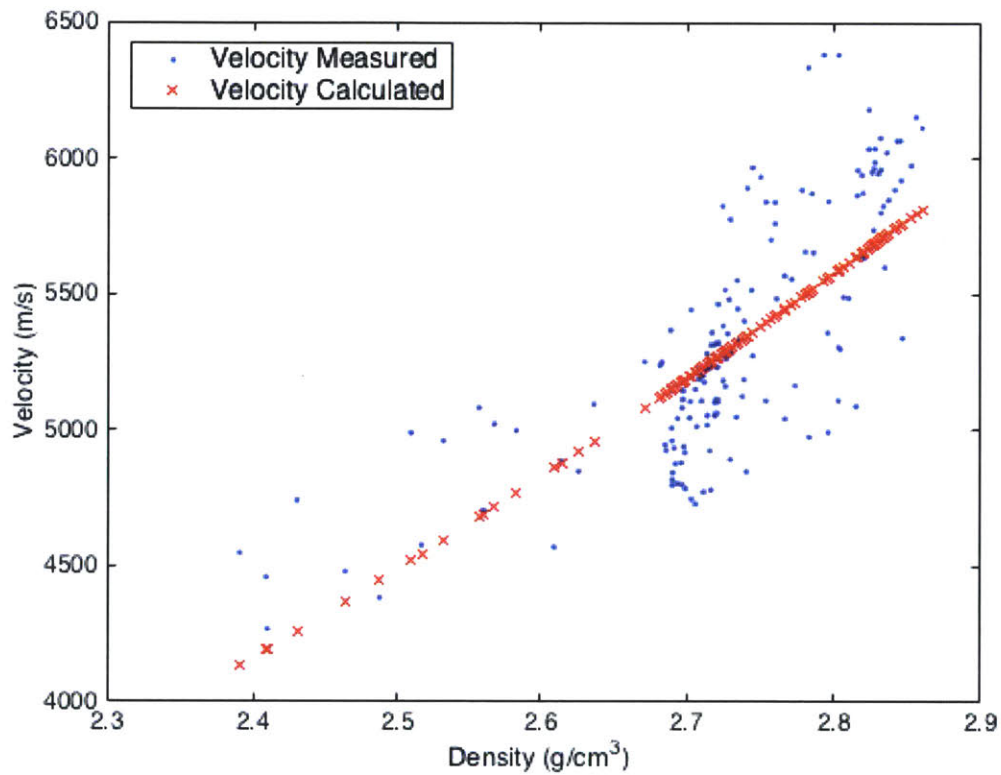


Figure 5-20: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Dolomite within Well No. 1 ($rmse = 0.05$)

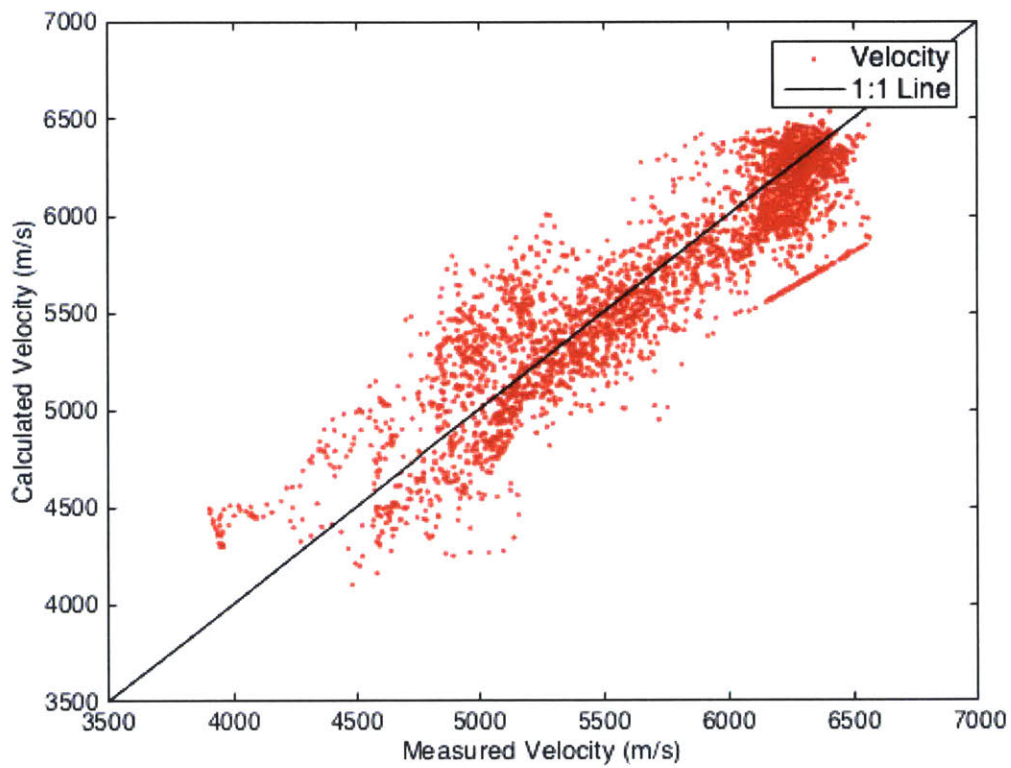


Figure 5-21: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Limestone within Well No. 2 ($rmse = 0.05$)

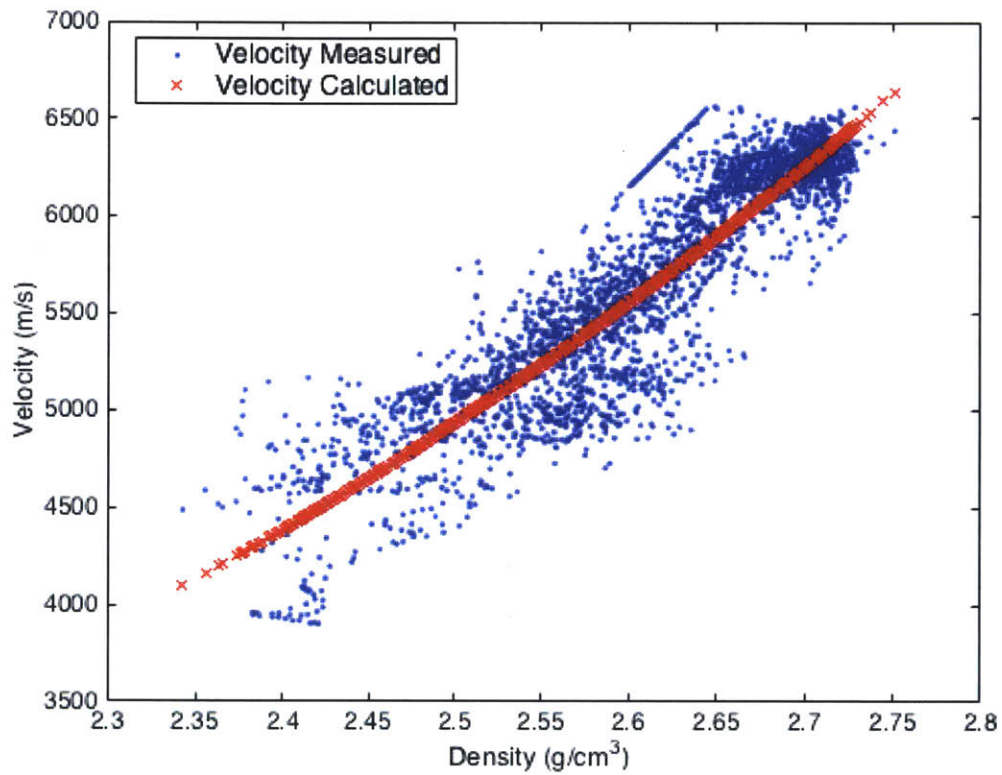


Figure 5-22: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Limestone within Well No. 2 ($rmse = 0.05$)

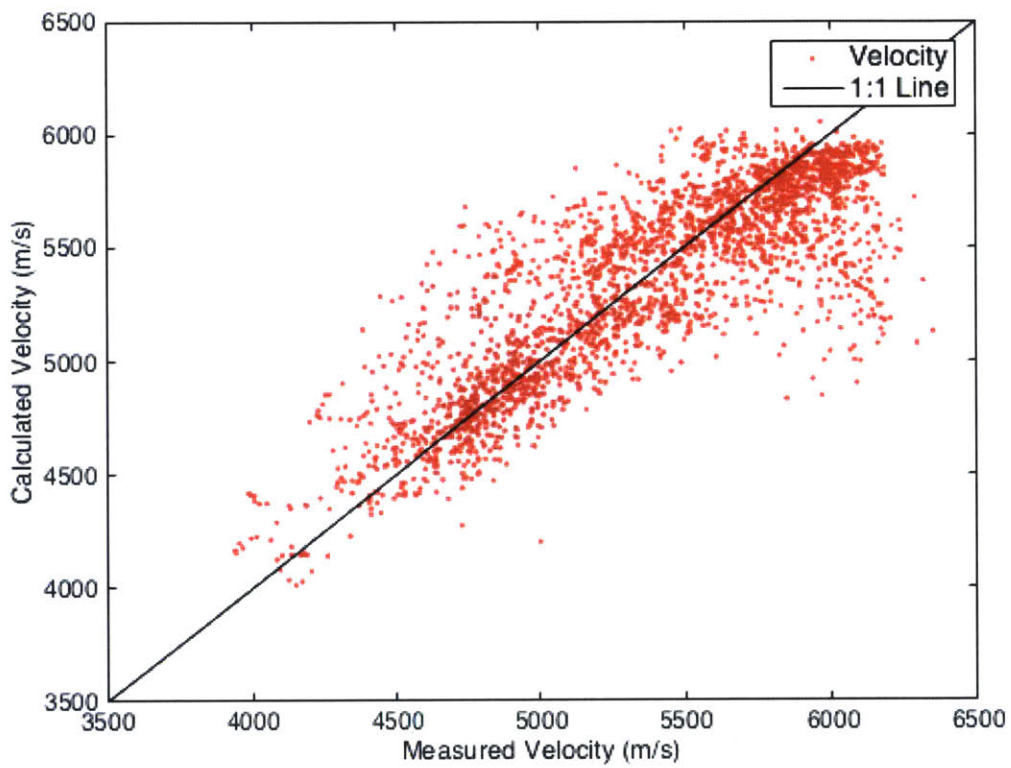


Figure 5-23: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] versus measured velocity for Limestone within Well No. 3 ($rmse = 0.05$)

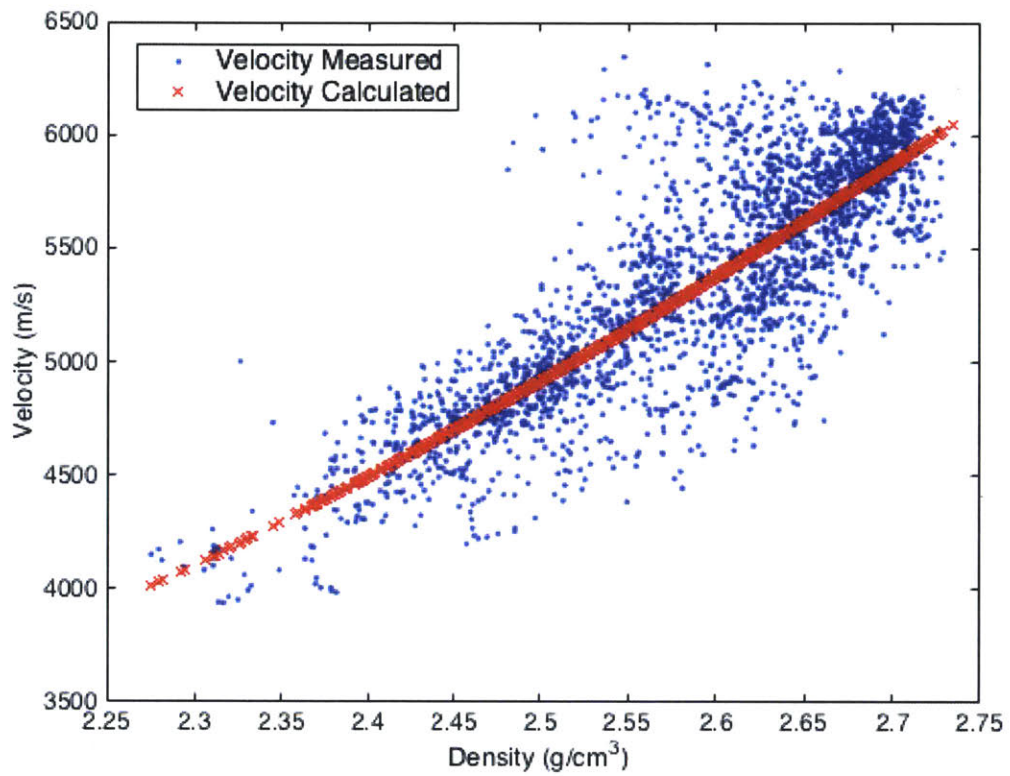


Figure 5-24: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Limestone within Well No. 3 ($rm.se = 0.05$)

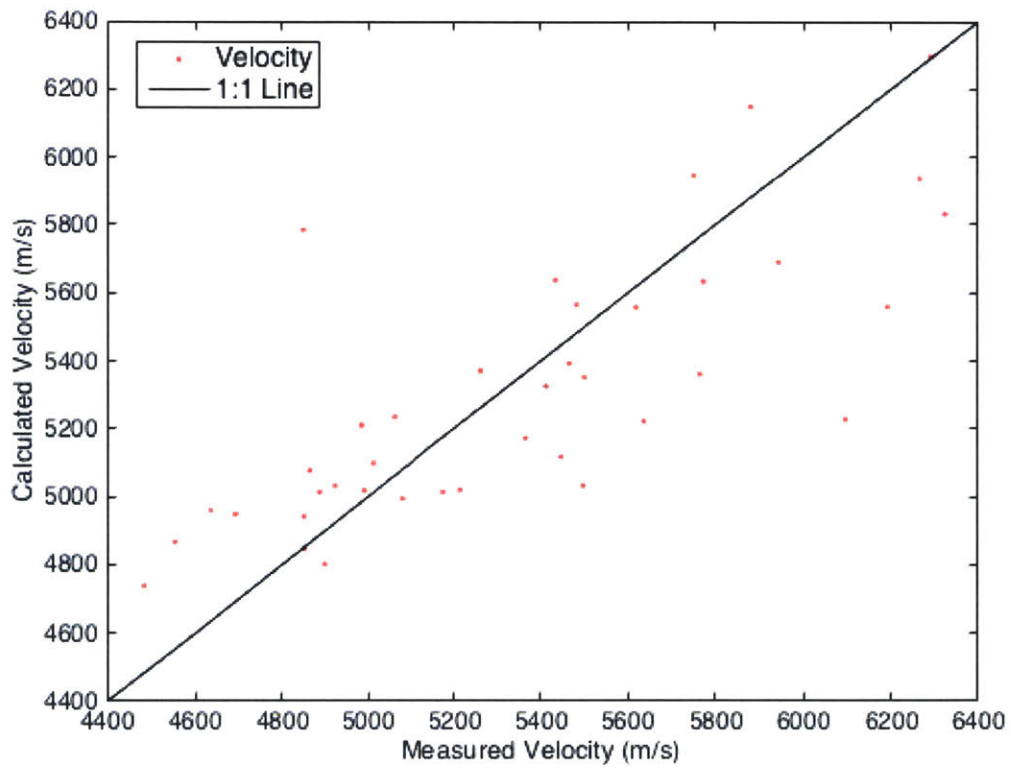


Figure 5-25: The calculated velocity using the generalized form of Gardner's equation $[V = a\rho^b + c]$ versus measured velocity for Dolomite within Well No. 3 ($rmse = 0.06$)

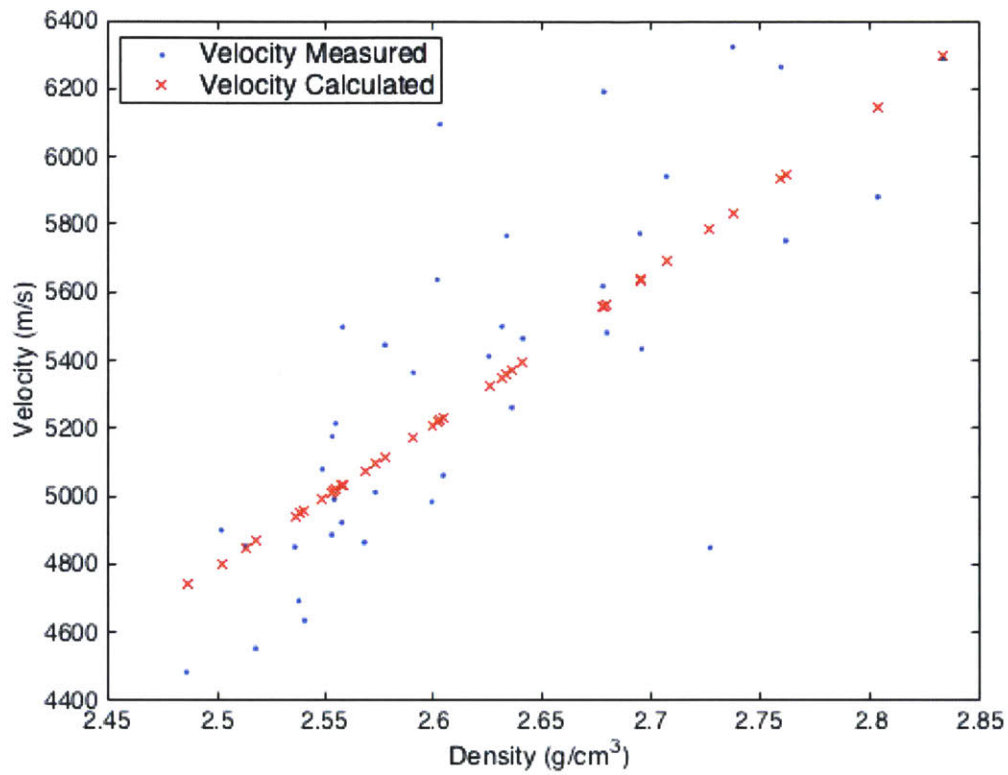


Figure 5-26: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Dolomite within Well No. 3 ($rmse = 0.06$)

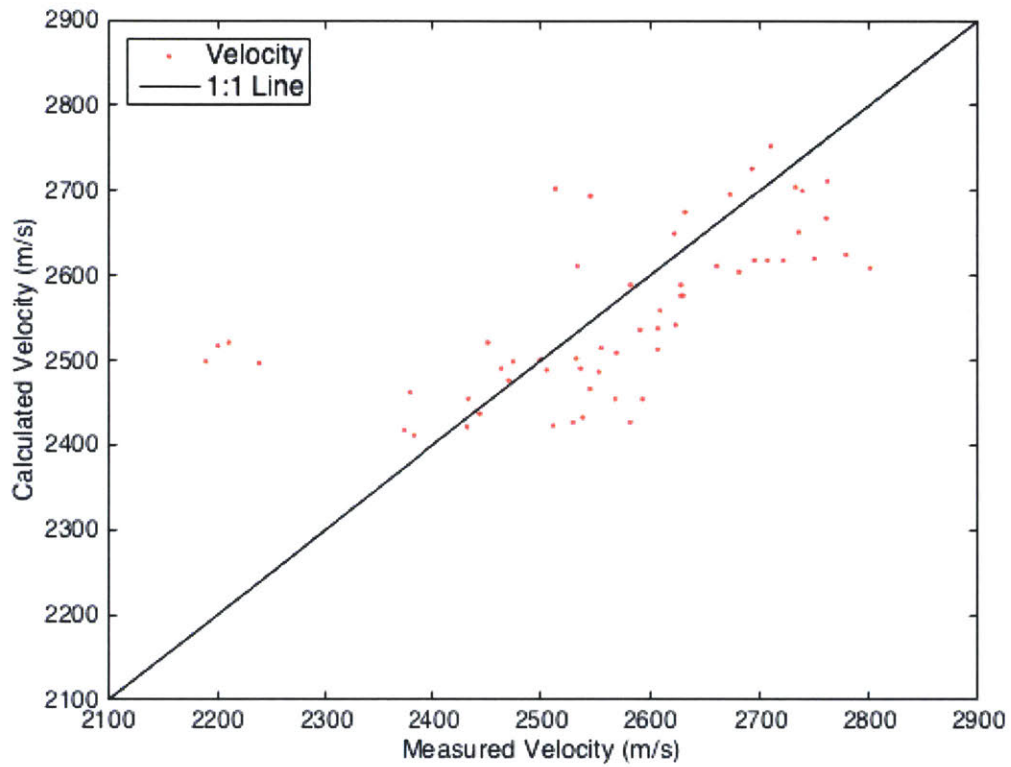


Figure 5-27: The calculated velocity using the generalized form of Gardner's equation $[V = a\rho^b + c]$ versus measured velocity for Clean Sands within Well No. 4 ($rmse = 0.05$)

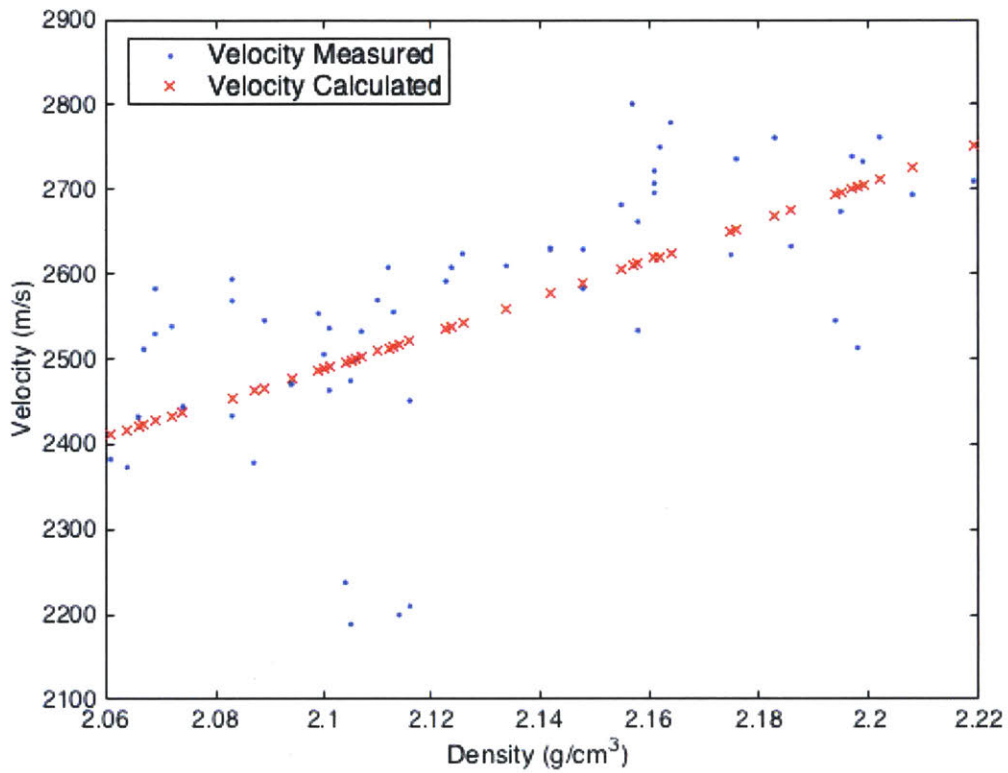


Figure 5-28: The calculated velocity using the generalized form of Gardner's equation [$V = a\rho^b + c$] and measured velocity versus density for Clean Sands within Well No. 4 ($rmse = 0.05$)

Chapter 6

Conclusion

This thesis investigated the inconsistency between theory and observations regarding the relationship between velocity and density of saturated porous rocks. In Chapter 3, it was concluded that the velocity behaves differently depending on the medium in which the wave is travelling. For gases and non-porous solids, which are considered single phase materials, the velocity obeys the Newton-Laplace equation. On the other hand, introducing another phase to the medium changes the wave behaviour, which is the case with porous saturated rocks. In addition, it was also confirmed that the velocities through different rocks depend on density to approximately the 4th power as already stated by Gardner (Gardner et al., 1974).

Because of the fact that the velocity calculated using Wyllie's equation increases with increasing density, a new expression for the elastic modulus was derived in Chapter 4 by linking the Newton-Laplace equation to Wyllie's equation. The new expression of the elastic modulus can be calculated using the parameters of the mineral, fluid, and the bulk density or porosity of the rock. In addition, it was also concluded that the elastic modulus depends numerically on density in the form ($M = A\rho^9 + B\rho^5 + C\rho$) or approximately ($M = a\rho^9$), as summarized in figure 6-1. Moreover, Gardner's equation was modified to insure that the velocity is correct over the entire range of densities ($V = a\rho^b + c$).

Chapter 5 showed the results of applying the findings in Chapter 4 using real well-log data from 4 wells. The new derived expression for the elastic modulus, M, provided

$$\frac{1}{V} = \frac{\phi}{V_f} + \frac{(1-\phi)}{V_m} \qquad V = \sqrt{\frac{M}{\rho}}$$

$$M = \frac{(\rho_m - \rho_f)^2 V_f^2 V_m^2 \rho}{((V_f - V_m) \rho + (\rho_m V_m - \rho_f V_f))^2}$$

$$M = A\rho^9 + B\rho^5 + C\rho$$

$$V = a\rho^4 + c$$

Figure 6-1: Linking the Newton-Laplace equation to Wyllie's equation to produce new expression for the elastic modulus, M, the approximate relationship between the elastic modulus and density, and the approximate relationship between velocity and density

results that are in good agreement with measured data. Moreover, a grid search is used to obtain the fit parameters of the generalized form of Gardner's equation to real data. In addition, the weighted average of the fit parameter "b" for all different lithologies was found to be 3.7, which is close the value of the fit parameter "b" for the original form of Gardner's equation and Wyllie's equation (4).

The velocity calculated using the new expression of the elastic modulus, M , (equation 4.15) is indistinguishable when compared to the the velocity calculated using the numerical approximation of velocity (equation 4.23), which was based on Gardner's approximation. In addition, both ways of calculating and approximating velocity provided small *rmse* errors.

Bibliography

- Avseth, P., Mukerji, T., and Mavko, G. (2005). *Quantitative Seismic Interpretation*. Cambridge University Press.
- Biot, M. A. (1956a). Theory of propagation of elastic waves in a fluid-saturated porous solid. i. higher-frequency range. *The Journal of The Acoustical society of America*, 28(2):179–191.
- Biot, M. A. (1956b). Theory of propagation of elastic waves in a fluid-saturated porous solid. i. low-frequency range. *The Journal of The Acoustical society of America*, 28(2):168–178.
- Bourvie, T., Coussy, O., and Zinszner, B. (1987). *The Acoustics of Porous Media*. Gulf Publishing Company.
- de Podesta, M. (2002). *Understanding the Properties of Matter*. Taylor Francis Group, second edition.
- Domenico, S. N. (1977). Elastic properties of unconsolidated porous sand reservoirs. *Geophysics*, 42(7):1339–1368.
- Finn, B. S. (1964). Laplace and the speed of sound. *ISIS*, 55(179):7–19.
- Gardner, G. H. F., Gardner, L. W., and Gregory, A. R. (1974). Formation velocity and density-the diagnostic basics for stratigraphic traps. *Geophysics*, 39(6):770–780.
- Gassman, F. (1951). Elastic waves through a packing of spheres. *Geophysics*, 15:673–685.
- Geertsma, J. and Smit, D. C. (1961). Some aspects of elastic wave propagation in fluid-saturated porous solids. *Geophysics*, 26(2):169–181.
- Gue'guen, Y. and Palciauskas, V. (1994). *Introduction to the Physics of Rocks*. Princeton University Press.
- Hashin, Z. and Shtrikman, S. (1963). A variational approach to the theory of the elastic behaviour of multiphase materials. *Journal of the Mechanics and Physics of Solids*, 11:127–140.
- Jones, R. M. (2009). *Deformation Theory of Plasticity*. Bull Ridge Publishing.

- Marion, D. P. (1990). *Acoustical, Mechanical, and Transport Properties of Sediments and Granular Materials*. PhD thesis, Stanford University.
- Mavko, G. (2016). Introduction to rock physics.
- Mavko, G., Mukerji, T., and Dvorkin, J. (2003). *The Rock Physics Handbook*. Cambridge University Press.
- Sheriff, R. E. and Geldart, L. P. (1995). *Exploration Seismology*. Cambridge University Press, second edition.
- White, J. E. (1965). *Seismic Waves: Radiation, Transmission, and Attenuation*. McGraw-Hill Book Company.
- Wyllie, M. R. J., Gregory, A. R., and Gardner, G. H. F. (1958). An experimental investigation of factors affecting elastic wave velocities in porous media. *Geophysics*, 23(3):459–493.
- Wyllie, M. R. J., Gregory, A. R., and Gardner, L. W. (1956). Elastic wave velocities in heterogeneous porous media. *Geophysics*, 21(1):41–70.