Economic Analysis of Product-Flexible Manufacturing System Investment Decisions

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Comments are welcome.

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ABSTRACT

This paper presents a model of the tradeoffs involved in investing in flexible manufacturing capacity. Our model assumes that the firm must make its investment decision in manufacturing capacity before the resolution of uncertainty in product demand. Flexible capacity provides to the firm the ability to be responsive to a wide variety of future demand outcomes. This benefit is weighed against the increased cost of flexible manufacturing capacity vis a vis dedicated or nonflexible capacity.

We provide a general formulation of the two-product version of this problem where the firm must choose a portfolio of flexible and nonflexible capacity before receiving final demand information. Under the assumption that demand curves are linear, our model is a two-stage convex quadratic program. Our results characterize the optimal profit function and the optimal investment policies. Furthermore, we present general sensitivity analysis results that show how changes in the acquisition costs of flexible and nonflexible capacity affect the optimal investment decisions. Finally, we present a numerical example where we compute the optimal investment decisions and the value of flexibility in the context of an automibile engine plant investment decision.

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Introduction

Advances in microprocessor-based manufacturing technologies have hastened the development of automated manufacturing systems that are being noted for their flexibility. Compared with the less automated systems they are designed to replace, flexible manufacturing sytems often cost more to acquire and install, but yield lower variable costs, higher product conformance (quality), and greater flexibility. Management analysts have a great deal of knowledge and experience in how to evaluate investments that reduce operating costs, a moderate amount of knowledge and experience in evaluating cost and quality tradeoffs, and much less knowledge and experience in evaluating investments that enhance flexibility.

The model in this paper contributes to the knowledge base on analyzing investments in manufacturing flexibility. The subject of the economics of flexibility has been of interest to economists for a long time (see, e.g., Stigler [1939], and the many references in Jones and Ostroy [1984]), but has become of significant interest in the OR/MS community only recently, following the increasing viability of flexible, computer-controlled manufacturing systems. This new interest has spurred a large amount of work in a very short time. (See, e.g., Stecke and Suri [1984], Adler [1985], Kulatilaka [1985]).

The fundamental structure of the flexibility investment decision problem is quite simple. The central issues are the timing of investment and production decisions and the resolution of uncertainty. Our model assumes that firms must invest in plant and equipment before they receive their final information on product demand. Thus, firms will prefer plant and equipment with manufacturing flexibility so that they can set production optimally once the final demand information becomes available. This desired flexibility can be obtained at the investment stage, but only at a cost. Herein lies the tension in the model (and in the real-life decision problem).

We provide a general formulation of the two-product version of this problem where the firm must choose a portfolio of flexible and nonflexible capacity before receiving final demand information. Our results characterize the optimal profit function and the optimal investment policies. Furthermore, we provide a sensitivity analysis to show how changes in the acquisition costs of flexible and nonflexible capacity affect the optimal investment decisions.

Our model focuses on investments in product-flexible manufacturing systems (PFMS). By a product-flexible manufacturing system, we mean one that can produce a number of different products with very low changeover costs and times. This definition of flexibility is consistent with Mandelbaum's [1978] and Buzacott's [1982] state flexibility and Browne's [1984] production flexibility. (See also Adler [1985] Section 4 for a review of the often confusing, non-standardized flexibility classifications and definitions).

In this paper, we concentrate on investments in PFMS under uncertainty. Even in a world of certainty, product-flexibility is valuable, because equipment downtime for changeovers is affected favorably by the use of flexible equipment. Tradeoffs of this sort are appropriately analyzed by comparing total costs of system operation after optimizing the scheduling rules for both flexible and nonflexible systems. In order to focus our paper at a more aggregate level, we assume away the scheduling complexities for PFMS. We think that these scheduling issues are very important, but this modelling choice allows us to concentrate on the effects of uncertainty in product demand and the (short-term) irreversibility of investment decisions on the optimal mix of flexible and nonflexible manufacturing capacity.

In the next section we present our model formulation and derive some basic properties of the model. Section 2 presents properties of the optimal value function. In Section 3 we examine the optimal investment levels for flexible and nonflexible technologies and provide a sensitivity analysis for these results. Section 4 presents a numerical example where we compute the optimal investment decisions and the value of flexibility in an automobile engine plant investment setting. Section 5 contains a discussion of extensions and implications of the model.

1. PFMS Model Formulation and Basic Properties of the Model.

We assume that the firm can produce two different products, A and B. Before producing, the firm must make capacity investment decisions. Three types of capacity are available: dedicated A-capacity, dedicated B-capacity, and flexible AB-capacity. We denote by K_A, K_B, and K_{AB}, the amounts of each of these types of capacity invested in by the firm. The per unit purchase costs for these capacity types are r_A, r_B, and r_{AB}, respectively. Before making production decisions, but after making capacity investment decisions, the firm observes a random variable that provides information about the state of the world for market demand for products A and B. We assume there are k states of the world and the probability that state i occurs is p_i, where $p_i > 0$ for all i and $\sum_i p_i = 1$. After observing state i, the firm chooses its production levels X_{Ai} and X_{Bi} of products A and B, subject to the capacity constraints imposed by the earlier investment decisions.

The firm faces a linear demand curve for each product. In state i, the price consumers pay for each unit of product A is $e_i - f_i X_{Ai}$. Similarly, the demand curve for product B is $g_i - h_i X_{Bi}$. For simplicity, we assume there is only one period in which the firm makes production decisions. Finally, we assume that the variable cost of production of a unit of product A (or product B) is the same whether a unit is produced with flexible or nonflexible capacity. Thus, without loss of generality, we assume that the variable costs of production are zero. The implications of relaxing this and other assumptions are discussed in section 5 of the paper.

With this notation, we can formulate the PFMS investment decision problem as PFM1:

$$\underset{\substack{K_A, K_B, K_{AB} \\ X_{Ai}, X_{Bi}, i=1,\ldots,k}}{\text{maximize}} \; -r_A K_A - r_B K_B - r_{AB} K_{AB} + \sum_{i=1}^{k} p_i (X_{Ai}(e_i - f_i X_{Ai}) + X_{Bi}(g_i - h_i X_{Bi}))$$

$$\begin{array}{llllll}
\text{subject to:} & X_{Ai} - K_A - K_{AB} & \leq 0 & i=1,\ldots,k & \text{(i)} & (\delta_i) \\
& X_{Bi} - K_B - K_{AB} & \leq 0 & i=1,\ldots,k & \text{(ii)} & (\theta_i) \\
& X_{Ai} + X_{Bi} - K_A - K_B - K_{AB} & \leq 0 & i=1,\ldots,k & \text{(iii)} & (\lambda_i) \\
& X_{Ai}, X_{Bi} & \geq 0 & i=1,\ldots,k & \text{(iv)} & (q_i, r_i) \\
& K_A \geq 0,\ K_B \geq 0,\ K_{AB} & \geq 0 & & \text{(v)} & (m,n,p)
\end{array}$$

The notation for the Lagrange multipliers for each of the inequalities (i)-(v) is given to the far right of each line. We make the following assumptions:

A1. $e_i > 0$, $g_i > 0$, i.e., the intercept of each demand curve is positive.

A2. f_i, $h_i > 0$, i.e., each product's demand curve is strictly downward sloping.

A3. $r_A > 0$, $r_B > 0$, $r_{AB} > 0$, i.e., there are positive purchase costs for all capacities.

A4. $r_{AB} > r_A$, $r_{AB} > r_B$, i.e. the per unit cost of flexible capacity strictly exceeds the per unit cost of nonflexible capacity for either product. (Note that if $r_{AB} \leq r_A$ and $r_{AB} \leq r_B$, then the PFMS problem simplifies significantly because no firm would ever purchase any nonflexible capacity.)

A5. $r_{AB} < r_A + r_B$, i.e. the per unit cost of flexible capacity is strictly less than the per unit cost of purchasing nonflexible capacity for both products. (Otherwise, flexible capacity is obviously never economical. When assumption A5 is violated, the PFMS problem splits into separate investment decision problems for each product, and each problem has a closed form solution, see Freund and Fine [1986].)

For notational convenience, we will substitute $b_i = p_i e_i$, $a_i = p_i f_i$, $d_i = p_i g_i$, $c_i = p_i h_i$, $i=1,\ldots,k$ in PFM1. Assumptions A1 and A2 then are equivalent to a_i, b_i, c_i, $d_i > 0$, $i=1,\ldots,k$.

Under assumptions A1-A5 the PFMS problem is a convex quadratic program. The Karush-Kuhn-Tucker (K-K-T) conditions ensure that $\bar{X}_{Ai}$, $\bar{X}_{Bi}$, $i=1,\ldots,k$, $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$ constitute an optimal solution to the PFMS problem if and only if there exists nonnegative multipliers m, n, p, δ_i, θ_i, λ_i, q_i, r_i, $i=1,\ldots,k$, for which the following conditions hold:

$$b_i - 2a_i\bar{X}_{Ai} = \delta_i + \lambda_i - q_i \qquad i=1,\ldots,k$$

$$d_i - 2c_iX_{Bi} = \theta_i + \lambda_i - r_i \qquad i=1,\ldots,k$$

$$r_A = \sum_{i=1}^{k} (\delta_i + \lambda_i) + m$$

$$r_B = \sum_{i=1}^{k} (\theta_i + \lambda_i) + n$$

$$r_{AB} = \sum_{i=1}^{k} (\delta_i + \theta_i + \lambda_i) + p \quad ,$$

together with the complementary slackness conditions: .

$$\delta_i(\bar{X}_{Ai} - \bar{K}_A - \bar{K}_{AB}) = 0 \qquad i=1,\ldots,k$$

$$\theta_i(\bar{X}_{Bi} - \bar{K}_A - \bar{K}_{AB}) = 0 \qquad i=1,\ldots,k$$

$$\lambda_i(\bar{X}_{Ai} + \bar{X}_{Bi} - \bar{K}_A - \bar{K}_B - \bar{K}_{AB}) = 0 \qquad i=1,\ldots,k$$

and $q_i\bar{X}_{Ai} = 0$, $r_i\bar{X}_{Bi} = 0$, $i=1,\ldots,k$, $m\bar{K}_A = 0$, $n\bar{K}_B = 0$, $p\bar{K}_{AB} = 0$.

Theorem 1. (Existence and Uniqueness of Solutions) Under assumptions A1-A5, the PFMS problem has a unique optimal solution $\bar{X}_{Ai}$, $\bar{X}_{Bi}$, $i=1,...,k$, $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$.

PROOF: The proof is straightforward and is presented in the Appendix.

Corollary 1: Under assumptions A1-A5, the optimal values $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$, and $\bar{X}_{Ai}$, $\bar{X}_{Bi}$, $i=1,...,k$ are piecewise-linear continuous functions of the capacity cost data r_A, r_B, and r_{AB}.

PROOF: From the theory of parametric quadratic programming (see, e.g., Van de Panne [1975]), the set of optimal solutions is a piecewise-linear upper semi-continuous mapping of the linear coefficients in the objective function. By Theorem 1, this mapping is single valued, and so is a piecewise linear function and is continuous. [X]

Assumptions A4 and A5 play a critical role in guaranteeing the uniqueness of the solution in the PFMS problem. The necessity of these assumptions is illustrated in the following example. Costs of nonflexible capacity are $r_A = r_B = 10$, and there are two future states, with data:

State i	a_i	b_i	c_i	d_i
1	1/2	60	1/2	30
2	1/2	30	1/2	60

The table below shows optimal solutions to the PFMS problem for three different values of r_{AB}. Figure 1 illustrates the optimal $\bar{K}_{AB}$ as a function of r_{AB}.

Solution #	r_{AB}	$\bar{K}_A$	$\bar{K}_B$	$\bar{K}_{AB}$	$\bar{X}_{A1}$	$\bar{X}_{B1}$	$\bar{X}_{A2}$	$\bar{X}_{B2}$	$\bar{\delta}_1$	$\bar{\theta}_1$	$\bar{\lambda}_1$	$\bar{\delta}_2$	$\bar{\theta}_2$	$\bar{\lambda}_2$
#1	10	0	0	80	55	25	25	55	0	0	5	0	0	5
#2	10	25	25	30	55	25	25	55	0	0	5	0	0	5
#3	15	27.5	27.5	25	52.5	27.5	27.5	52.5	5	0	2.5	0	5	2.5
#4	20	30	30	20	50	30	30	50	10	0	0	0	10	0
#5	20	50	50	0	50	30	30	50	10	0	0	0	10	0

Upon setting $\bar{q}_1 = \bar{q}_2 = \bar{r}_1 = \bar{r}_2 = \bar{m} = \bar{n} = \bar{p} = 0$, it is easily verified that the solutions above satisfy the K-K-T conditions, and so are optimal. When $r_{AB} = 10$, $r_{AB} = r_A = r_B$, violating assumption A4. Solutions #1 and #2 are each optimal in this state. The firm is indifferent between building a mix of flexible and nonflexible capacity (solution #2) and building all flexible capacity (solution #1). When $r_{AB} = 20$, $r_{AB} = r_A + r_B$, violating assumption A5. Solutions #4 and #5 are each optimal. In this case, the firm is indifferent between building a mix of flexible and nonflexible capcity (solution #4), and building all nonflexible capacity (solution #5). Figure 1 shows the optimal values of $\bar{K}_{AB}$ as a function of r_{AB} in this example. Note that the optimal values of $\bar{K}_{AB}$ is a piecewise linear point-to-set mapping, and is a continuous function over the range $10 < r_{AB} < 20$, i.e., $r_A = r_B < r_{AB} < r_A + r_B$.

We now turn our attention to sensitivity analysis and comparative statics for the PFMS problem. Our analysis is facilitated by transforming the problem. For given production levels X_{Ai}, X_{Bi}, these quantities can be split into $X_{Ai} = Y_{Ai} + Z_{Ai}$, $X_{Bi} = Y_{Bi} + Z_{Bi}$, where Y_{Ai} is that portion of total A production being produced with nonflexible A-capacity, and Z_{Ai} is that portion

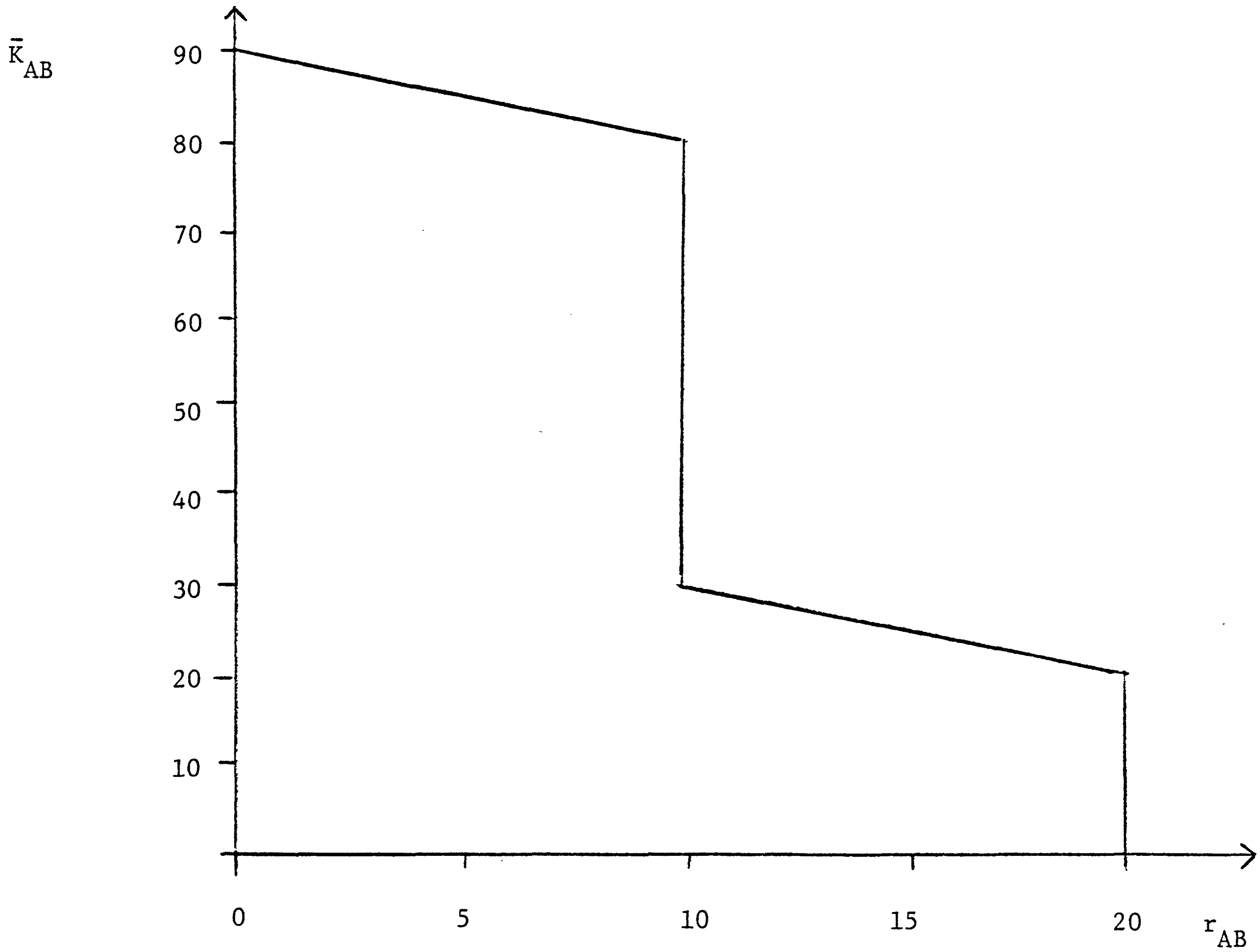

Figure 1. Optimal values of $\bar{K}_{AB}$ as a function of r_{AB}

of A production being produced with flexible AB-capacity (and similarly for B production). With this change of variables, the PFMS problem now becomes

PFM2:

$$\underset{\substack{K_A,\ K_B,\ K_{AB} \\ Y_{Ai},\ Z_{Ai},\ Y_{Bi},\ Z_{Bi} \\ i=1,\ldots,k}}{\text{maximize}} - r_A K_A - r_B K_B - r_{AB} K_{AB} + \sum_{i=1}^{k} - [a_i (Y_{Ai}+Z_{Ai})^2 + b_i (Y_{Ai}+Z_{Ai})] + \sum_{i=1}^{k} - [c_i (Y_{Bi}+Z_{Bi})^2 + d_i (Y_{Bi}+Z_{Bi})]$$

subject to:

$$\begin{array}{llll} Y_{Ai} - K_A \le 0 & i=1,\ldots,k & (i) & (\alpha_i) \\ Y_{Bi} - K_B \le 0 & i=1,\ldots,k & (ii) & (\beta_i) \\ Z_{Ai} + Z_{Bi} - K_{AB} \le 0 & i=1,\ldots,k & (iii) & (\gamma_i) \\ Y_{Ai} \ge 0 & i=1,\ldots,k & (iv) & (s_i) \\ Y_{Bi} \ge 0 & i=1,\ldots,k & (v) & (t_i) \\ Z_{Ai} \ge 0 & i=1,\ldots,k & (vi) & (u_i) \\ Z_{Bi} \ge 0 & i=1,\ldots,k & (vii) & (v_i) \\ K_A,\ K_B,\ K_{AB} \ge 0 & & (viii) & (m,n,p) \end{array}$$

It is a straightforward exercise to verify that PFM2 is equivalent to PFM1, with the transformations:

$$X_{Ai} = Y_{Ai} + Z_{Ai} \quad \text{and} \quad Y_{Ai} = \min (X_{Ai},\ K_A),\ Z_{Ai} = \max (X_{Ai} - K_A,\ 0)$$
$$X_{bi} = Y_{Bi} + Z_{Bi} \qquad Y_{Bi} = \min (X_{Bi},\ K_B),\ Z_{Bi} = \max (X_{Bi} - K_B,\ 0)$$
$$i=1,\ldots,k.$$

Theorem 1 asserts that under assumptions A1-A5, the quantities $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$, and $(\bar{Y}_{Ai} + \bar{Z}_{Ai})$, $(\bar{Y}_{Bi} + \bar{Z}_{Bi})$, $i=1,\ldots,k$ are uniquely determined in an optimal solution.

In the program PFM2, the strategic variables K_A, K_B, and K_{AB} are decoupled and appear in separate constraints. Furthermore, the shadow prices α_i, β_i, and γ_i are easier to interpret. For example, α_i is the marginal value of an extra unit of A-capacity in state i. Therefore, we would expect that at an optimal solution, $\sum_{i=1}^{k} \alpha_i = r_A$, equating marginal value and marginal cost of A-capacity. Similarly, we would expect $\sum_{i=1}^{k} \beta_i = r_B$ and $\sum_{i=1}^{k} \gamma_i = r_{AB}$. Indeed, the optimality conditions for the problem bear out this intuition. The K-K-T conditions for PFM2 ensure that a feasible solution $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$, $\bar{Y}_{Ai}$, $\bar{Z}_{Ai}$, $\bar{Y}_{Bi}$, $\bar{Z}_{Bi}$, i=1,...,k, is an optimal solution to PFM2 if and only if there exist nonnegative multipliers $\bar{m}$, $\bar{n}$, $\bar{p}$, $\bar{\alpha}_i$, $\bar{\beta}_i$, $\bar{\gamma}_i$, $\bar{s}_i$, $\bar{t}_i$, $\bar{u}_i$, $\bar{v}_i$, i=1,...,k, for which the following conditions hold:

(I) $b_i - 2a_i(\bar{Y}_{Ai} + \bar{Z}_{Ai}) = \bar{\alpha}_i - \bar{s}_i \quad i=1,\dots,k$

(II) $b_i - 2a_i(\bar{Y}_{Ai} + \bar{Z}_{Ai}) = \bar{\gamma}_i - \bar{u}_i \quad i=1,\dots,k$

(III) $d_i - 2c_i(\bar{Y}_{Bi} + \bar{Z}_{Bi}) = \bar{\beta}_i - \bar{t}_i \quad i=1,\dots,k$

(IV) $d_i - 2c_i(\bar{Y}_{Bi} + \bar{Z}_{Bi}) = \bar{\gamma}_i - \bar{v}_i \quad i=1,\dots,k$

(V) $r_A = \sum_{i=1}^{k} \bar{\alpha}_i + \bar{m}$

(VI) $r_B = \sum_{i=1}^{k} \bar{\beta}_i + \bar{n}$

(VII) $r_{AB} = \sum_{i=1}^{k} \bar{\gamma}_i + \bar{p}$

(VIII) $\bar{\alpha}_i \; (\bar{K}_A - \bar{Y}_{Ai}) = 0$ $\quad i=1,\ldots,k$

$\bar{\beta}_i \; (\bar{K}_B - \bar{Y}_{Bi}) = 0$ $\quad i=1,\ldots,k$

$\bar{\gamma}_i \; (\bar{K}_{AB} - \bar{Z}_{Ai} - \bar{Z}_{Bi}) = 0$ $\quad i=1,\ldots,k$

(IX) $\bar{s}_i \; \bar{Y}_{Ai} = 0, \; \bar{t}_i \; \bar{Y}_{Bi} = 0, \; \bar{u}_i \; \bar{Z}_{Ai} = 0, \; \bar{v}_i \; \bar{Z}_{Bi} = 0, \; i=1,\ldots,k,$

$\bar{m} \; \bar{K}_A = 0, \; \bar{n} \; \bar{K}_B = 0, \; \bar{p} \; \bar{K}_{AB} = 0.$

Note that s_i, t_i, u_i, v_i, m, n, and p represent shadow prices on the nonnegativity conditions (iv) - (viii) of the problem PFM2. If production with each capacity is positive in each state, then by the complementary slackness conditions (IX), $\bar{s}_i = \bar{t}_i = \bar{u}_i = \bar{v}_i = \bar{m} = \bar{n} = \bar{p} = 0$, $i=1,\ldots,k$. Condition (V) states that the sum of the marginal contributions of nonflexible capacity for product A must be less than or equal to the cost of nonflexible A-capacity, with equality holding when A-capacity is purchased, (by (IX)). Conditions (VI) and (VII) are interpreted in similar fashion for nonflexible B-capacity, and for flexible AB-capacity.

The PFMS problem has a natural interpretation as a two-stage program, where the capacity variables K_A, K_B, K_{AB} are the first-stage variables, and the production variables Y_{Ai}, Z_{Ai}, Y_{Bi}, Z_{Bi} are the second-stage variables. Given optimal values $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$ for K_A, K_B, and K_{AB}, in the PFMS problem, the k independent second-stage programs P(i) are:

$$\begin{array}{ll} \text{maximize} & -a_i (Y_{Ai} + Z_{Ai})^2 + b_i (Y_{Ai} + Z_{Ai}) \\ Y_{Ai}, Z_{Ai}, Y_{Bi}, Z_{Bi} & -c_i (Y_{Bi} + Z_{Bi})^2 + d_i (Y_{Bi} + Z_{Bi}) \end{array}$$

subject to:

$$\begin{array}{rl} Y_{Ai} \leq \bar{K}_A & (\alpha_i) \\ Y_{Bi} \leq \bar{K}_B & (\beta_i) \\ Z_{Ai} + Z_{Bi} \leq \bar{K}_{AB} & (\gamma_i) \\ Y_{Ai} \geq 0 & (s_i) \\ Y_{Bi} \geq 0 & (t_i) \\ Z_{Ai} \geq 0 & (u_i) \\ Z_{Bi} \geq 0 & (v_i) \end{array}$$

An optimal solution $\bar{Y}_{Ai}$, $\bar{Z}_{Ai}$, $\bar{Y}_{Bi}$, $\bar{Z}_{Bi}$ to P(i), i=1,...,k, together with $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$ will constitute an optimal solution to the PFMS problem, and under assumptions A1-A5, the total production quantity $(\bar{Y}_{Ai} + \bar{Z}_{Ai})$ in program P(i) will be unique.

For notational convenience, let $\bar{K} = (\bar{K}_A, \bar{K}_B, \bar{K}_{AB})$ represent the vector of optimal capacity values. By $\bar{K} > 0$ we mean $\bar{K}_A > 0$, $\bar{K}_B > 0$, and $\bar{K}_{AB} > 0$.

Lemma 1. Let $\bar{K}$ be optimal capacity levels for the PFMS problem satisfying assumptions A1-A5, and let $\bar{Y}_{Ai}$, $\bar{Z}_{Ai}$, $\bar{Y}_{Bi}$, $\bar{Z}_{Bi}$, be optimal solutions to P(i), i=1,...,k. Then the set of multipliers defined in (1) below is an optimal set of K-K-T multipliers for the PFMS problem. Furthermore, if $\bar{K} > 0$, then these K-K-T multipliers are the unique K-K-T multipliers for the PFMS problem.

$$
\begin{aligned}
\bar{\alpha}_i &= [b_i - a_i\,(\bar{Y}_{Ai} + \bar{Z}_{Ai})]^+ \,, \\
\bar{s}_i &= [b_i - a_i\,(\bar{Y}_{Ai} + \bar{Z}_{Ai})]^- \,, \\
\bar{\beta}_i &= [d_i - c_i\,(\bar{Y}_{Bi} + \bar{Z}_{Bi})]^+ \,, \\
\bar{t}_i &= [d_i - c_i\,(\bar{Y}_{Bi} + \bar{Z}_{Bi})]^- \,, \\
\bar{\gamma}_i &= \max\,(\bar{\alpha}_i, \bar{\beta}_i) \,, \\
\bar{u}_i &= \bar{\gamma}_i - \bar{\alpha}_i + \bar{s}_i \,, \qquad\qquad (1) \\
\bar{v}_i &= \bar{\gamma}_i - \bar{\beta}_i + \bar{t}_i \,, \quad i=1,\ldots,k. \\
\bar{m} &= r_A - \sum_{i=1}^{k} \bar{\alpha}_i \\
\bar{n} &= r_B - \sum_{i=1}^{k} \bar{\beta}_i \\
\bar{p} &= r_{AB} - \sum_{i=1}^{k} \bar{\gamma}_i \,.
\end{aligned}
$$

<u>PROOF</u>: The quantities $\bar{Y}_{Ai}$, $\bar{Z}_{Ai}$, $\bar{Y}_{Bi}$, $\bar{Z}_{Bi}$ are an optimal solution to P(i) if and only if they form a solution to the PFMS problem together with the optimal capacity levels $\bar{K}_A$, $\bar{K}_B$, and $\bar{K}_{AB}$. By Theorem 1, the quantities $(\bar{Y}_{Ai} + \bar{Z}_{Ai}) = \bar{X}_{Ai}$ and $(\bar{Y}_{Bi} + \bar{Z}_{Bi}) = \bar{X}_{Bi}$ are uniquely determined, and so the multipliers defined in the theorem are uniquely specified by the formulas as functions of $\bar{Y}_{Ai} + \bar{Z}_{Ai}$ and $\bar{Y}_{Bi} + \bar{Z}_{Bi}$, $i=1,\ldots,k$. All multipliers are obviously nonnegative with the possible exception of $\bar{m}$, $\bar{n}$, and $\bar{p}$. Furthermore, these multipliers satisfy (I) - (VII) of the K-K-T conditions for the PFMS problem. It thus remains to show that the complementarity conditions (IX) of the K-K-T conditions are met, as well as that $\bar{m}$, $\bar{n}$, $\bar{p}$ are nonnegative.

If $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\tilde{\gamma}_i$, $\tilde{s}_i$, $\tilde{t}_i$, $\tilde{u}_i$, $\tilde{v}_i$, $i=1,\ldots,k$, $\tilde{m}$, $\tilde{n}$, $\tilde{p}$ are a set of optimal K-K-T multipliers, then from the uniqueness of

$(\bar{Y}_{Ai} + \bar{Z}_{Ai})$, we must have that $\tilde{\alpha}_i - \tilde{s}_i = \bar{\alpha}_i - \bar{s}_i$. But $\bar{\alpha}_i \cdot \bar{s}_i = 0$ and all quantities must be nonnegative, whereby $\tilde{\alpha}_i \geq \bar{\alpha}_i$, $\tilde{s}_i \geq \bar{s}_i$, $i=1,\ldots,k$. Similarly, we can show that $\tilde{\beta}_i \geq \bar{\beta}_i$, $\tilde{t}_i \geq \bar{t}_i$, $\tilde{\gamma}_i \geq \bar{\gamma}_i$, $\tilde{u}_i \geq \bar{u}_i$, $\tilde{v}_i \geq \bar{v}_i$, $i=1,\ldots,k$. Thus, because $\tilde{s}_i \bar{Y}_{Ai} = 0$, then $\bar{s}_i \bar{Y}_{Ai} = 0$, and similarly $\bar{t}_i \bar{Y}_{Bi} = 0$, $\bar{u}_i \bar{Z}_{Ai} = 0$, $\bar{v}_i \bar{Z}_{Bi} = 0$. Also $\bar{m} = r_A - \sum^k \bar{\alpha}_i \geq r_A - \sum^k \tilde{\alpha}_i = \tilde{m} \geq 0$, and so $\bar{m} \geq 0$ and $\bar{m} \geq \tilde{m}$, and so $\bar{m} \bar{K}_A = 0$ implies that $\tilde{m} \bar{K}_A = 0$. Similarly, $\bar{n} \geq 0$, $\bar{p} \geq 0$, and $\bar{n} \bar{K}_B = 0$ and $\bar{p} \bar{K}_{AB} = 0$. Thus the multipliers specified in the theorem are optimal K-K-T multipliers, and indeed are the smallest values that any set of multipliers can take.

If $\bar{K} > 0$, then $\bar{m} = 0$, and so $\sum_{i=1}^{k} \bar{\alpha}_i = \sum_{i=1}^{k} \tilde{\alpha}_i = r_{AB}$. Thus, since $\bar{\alpha}_i \leq \tilde{\alpha}_i$, $\bar{\alpha}_i = \tilde{\alpha}_i$, $i=1,\ldots,k$. Similarly, $\bar{\beta}_i = \tilde{\beta}_i$, $\bar{\gamma}_i = \tilde{\gamma}_i$, and it then follows that $\bar{s}_i = \tilde{s}_i$, $\bar{t}_i = \tilde{t}_i$, $\bar{u}_i = \tilde{u}_i$, $\bar{v}_i = \tilde{v}_i$, $i=1,\ldots,k$, and $\bar{m} = \tilde{m} = \bar{n} = \tilde{n} = \bar{p} = \tilde{p} = 0$, establishing the the uniqueness of the optimal K-K-T multipliers in this case. [X]

<u>Corollary 2</u>. There exists a unique set of optimal K-K-T multipliers that satisfy $\bar{\gamma}_i = \max(\bar{\alpha}_i, \bar{\beta}_i)$, $i=1,\ldots,k$. [X]

Note that, in particular, if $\bar{K} > 0$, then the optimal K-K-T multipliers are unique and hence must satisfy $\bar{\gamma}_i = \max(\bar{\alpha}_i, \bar{\beta}_i)$, $i=1,\ldots,k$.

The economic significance of this corollary is clear. The $\bar{\alpha}_i$ and $\bar{\beta}_i$ represent the marginal values of A-capacity and B-capacity, respectively, in state i, $i=1,\ldots,k$. Furthermore, $\bar{\gamma}_i$ represents the marginal value of AB-capacity in state i, $i=1,\ldots,k$. The condition

$\bar{\gamma}_i = \max(\bar{\alpha}_i, \bar{\beta}_i)$, $i=1,\dots,k$ asserts that in each state the marginal value of AB-capacity is the larger of the marginal value of A-capacity or of B-capacity.

In the absence of flexible capacity, the firm faces the decision of how much A-capacity and B-capacity to build by solving the two independent manufacturing investment problems AMS and BMS:

AMS:

$$\max_{\substack{K_A, X_{Ai} \\ i=1,\dots,k}} \; - r_A K_A + \sum_{i=1}^{k} [-a_i(X_{Ai})^2 + b_i X_{Ai}]$$

subject to: $X_{Ai} - K_A \leq 0 \quad (\alpha_i)$

$X_{Ai} \geq 0 \quad (s_i)$

$K_A \geq 0 \quad (m)$

BMS:

$$\max_{\substack{K_B, X_{Bi} \\ i=1,\dots,k}} \; - r_B K_B + \sum_{i=1}^{k} [-c_i(X_{Bi})^2 + d_i X_{Bi}]$$

subject to: $X_{Bi} - K_B \leq 0 \quad (\beta_i)$

$K_{Bi} \geq 0 \quad (t_i)$

$K_B \geq 0 \quad (n)$

As is discussed in a companion paper (Freund and Fine [1986]), these two problems possess unique solutions and unique K-K-T multipliers, and can be solved analytically. In view of corollary 2, we have the following theorem:

Theorem 2 (Necessary and sufficient conditions for purchasing flexible capacity).

Let $\bar{K}_A$, $\bar{X}_{Ai}$, $\bar{\alpha}_i$, $\bar{s}_i$, $i=1,\dots,k$, $\bar{m}$, and $\bar{K}_B$, $\bar{X}_{Bi}$, $\bar{\beta}_i$, $\bar{t}_i$, $i=1,\dots,k$, $\bar{n}$ be optimal solutions and K-K-T multipliers for the independent manufacturing investment problems AMS and BMS, under assumptions A1-A5.

Then $\bar{K}_{AB} > 0$ in the optimal solution to the PFMS problem if and only if $r_{AB} < \sum_{i=1}^{k} \max(\bar{\alpha}_i, \bar{\beta}_i)$.

The economic interpretation of Theorem 2 should be clear. In the respective problems AMS and BMS, $\bar{\alpha}_i$ and $\bar{\beta}_i$ represent the marginal values of extra A- and B-capacity in state i. Thus $\max(\bar{\alpha}_i, \bar{\beta}_i)$ represents the marginal value of capacity that can be used in production of either A or B, i.e., the marginal value of flexible capacity. The theorem states that in order for flexible capacity to be economical, its cost r_{AB} must be less than the sum, over all states, of the marginal value of the capacity's most valuable use in that state.

PROOF: If $r_{AB} \geq \sum_{i=1}^{k} \max(\bar{\alpha}_i, \bar{\beta}_i)$, define $\bar{\gamma}_i \equiv \max(\bar{\alpha}_i, \bar{\beta}_i)$,

$\bar{u}_i \equiv \bar{\gamma}_i - \bar{\alpha}_i + \bar{s}_i$, $\bar{v}_i \equiv \bar{\gamma}_i - \bar{\beta}_i + \bar{t}_i$, and $\bar{p} \equiv r_{AB} - \sum_{i=1}^{k} \bar{\gamma}_i$.

Then the K-K-T conditions for the PFMS problem are satisfied with $\bar{K}_{AB} = 0$.

On the other hand, if $r_{AB} < \sum_{i=1}^{k} \max(\alpha_i, \beta_i)$, then the above solution is the unique solution to the K-K-T conditions satisfying (1). However, $(r_{AB} - \sum_{i=1}^{k} \bar{\gamma}_i) < 0$, and so by lemma 1, the solution $\bar{K}_A$, $\bar{K}_B$, $\bar{X}_{Ai}$, $\bar{X}_{Bi}$, i=1,...,k, and $\bar{K}_{AB} = 0$ cannot be optimal. [X]

2. Properties of the Optimal Value Function

We now turn our attention to the sensitivity of the PFMS problem to the costs of capacity, namely r_A, r_B, and r_{AB}. Let $z^*(r_A, r_B, r_{AB})$ be the optimal value function of the PFMS problem for given capacity costs r_A, r_B, and r_{AB}. Our concern with

$z^*(r_A, r_B, r_{AB})$ lies in ascertaining the properties of this optimal value function, to predict the savings or costs due to decreased or increased capacity costs.

Let $r = (r_A, r_B, r_{AB})$ be the vector of capacity costs. Let $\Delta r = (\Delta r_A, \Delta r_B, \Delta r_{AB})$ denote a vector of changes in optimal capacity costs. Then $z^*(r+\Delta r) = z^*(r_A + \Delta r_A, r_B + \Delta r_B, r_{AB} + \Delta r_{AB})$ measures the optimal value of the PFMS problem when capacity costs are $r + \Delta r$.

Theorem 3 (Characterization of the Optimal Value Function)

Under assumptions A1-A5,

(i) $z^*(r_A, r_B, r_{AB})$ is a convex piecewise-quadratic function, with continuous first partial derivatives.

(ii) If $\bar{K} = (\bar{K}_A, \bar{K}_B, \bar{K}_{AB})$ are optimal capacity values when capacity costs are $r = (r_A, r_B, r_{AB})$, then

$$\frac{\partial z^*}{\partial r_A} = -\bar{K}_A$$

$$\frac{\partial z^*}{\partial r_B} = -\bar{K}_B$$

$$\frac{\partial z^*}{\partial r_{AB}} = -\bar{K}_{AB} \quad .$$

(iii) If $r = (r_A, r_B, r_{AB})$ is in the interior of a region where $z^*(r)$ is a quadratic form, and Δr is sufficiently small, then $z^*(r+\Delta r) = z^*(r)-\bar{K}_A(\Delta r_A)-\bar{K}_B(\Delta r_B)-\bar{K}_{AB}(\Delta r_{AB}) - (1/2)(\Delta r)^T M(\Delta r)$, where M is a symmetric negative semi-definite matrix. If $\bar{K} > 0$, then M is negative definite, and so $z^*(r+\Delta r)$ is strictly convex.

This theorem gives us the structure of the optimal value function - it is convex and piecewise-quadratic. Furthermore, it has continuous first partial derivatives given by (ii) in Theorem 3. The interpretation of these first partial derivatives is that profit declines with increases in capacity costs at a rate equal to the optimal capacity value. The effects of small changes in capacity costs on _optimal_ capacity levels and consequently on profits are of second-order. The theorem also shows that the optimal value function is decreasing in each capacity cost, because $\bar{K}_A \geq 0$, $\bar{K}_B \geq 0$, and and $\bar{K}_{AB} \geq 0$. The formula in (iii) gives us an explicit functional form for $z^*(r+\Delta r)$ if we know the matrix M. In the next section, as part of the proof of Theorem 4, we will show how to construct the matrix M, and hence how to construct the functional form in part (iii) of the above theorem.

PROOF: The proof of this theorem follows from the duality properties of convex quadratic programming, see Dorn [1960]. The standard quadratic programming dual of the PFMS problem PFM2, which we denote as the DFMS problem, can be derived as:

$$\underset{\substack{\alpha_i,\beta_i,\gamma_i\\ s_i,t_i\\ i=1,\dots,k}}{\text{minimize}} \quad \sum_{i=1}^{k} (b_i-\alpha_i+s_i)^2/(4a_i) + \sum_{i=1}^{k} (d_i-\beta_i+t_i)^2/(4c_i)$$

$$\text{subject to:} \quad \sum_{i=1}^{k} \alpha_i \leq r_A \qquad (K_A)$$

$$\sum_{i=1}^{k} \beta_i \leq r_B \qquad (K_B)$$

$$\sum_{i=1}^{k} \gamma_i \leq r_{AB} \qquad (K_{AB})$$

$$\alpha_i - s_i - \gamma_i \le 0 \quad , \ i=1,\dots,k \quad (z_{Ai})$$

$$\beta_i - t_i - \gamma_i \le 0 \quad , \ i=1,\dots,k \quad (z_{Bi})$$

$$s_i \ge 0 \quad , \ i=1,\dots,k \quad (\gamma_{Ai})$$

$$t_i \ge 0 \quad , \ i=1,\dots,k \quad (\gamma_{Bi})$$

$$\alpha_i \ge 0 \quad , \ i=1,\dots,k$$

$$\beta_i \ge 0 \quad , \ i=1,\dots,k$$

$$\gamma_i \ge 0 \quad , \ i=1,\dots,k.$$

where the primal variables corresponding the dual constraints have been written in to the right next to the appropriate dual constraint. Note that the capacity costs r_A, r_B, and r_{AB} now appear as RHS coefficients of dual constraints whose optimal K-K-T multipliers $\bar{K}_A$, $\bar{K}_B$, and $\bar{K}_{AB}$ will form a solution to the primal problem. From the theory of Lagrange duality, the optimal value function $z^*(r_A, r_B, r_{AB})$ is a convex function. Also, from Lagrange duality, $(\bar{K}_A, \bar{K}_B, \bar{K}_{AB})$ are optimal K-K-T multipliers for the first three constraints of the DFMS problem if and only if $(-\bar{K}_A, -\bar{K}_B, -\bar{K}_{AB})$ is a subgradient of $z^*(r_A, r_B, r_{AB})$, see Geoffrion [1971]. However, by Theorem 1, $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$ are uniquely determined, so $(-\bar{K}_A, -\bar{K}_B, -\bar{K}_{AB})$ is the gradient of $z^*(r_A, r_B, r_{AB})$, and each component is a partial derivative, obtaining formulas (ii) of the theorem. From corollary 1, these first partial derivatives are continuous in r_A, r_B, and r_{AB}. Finally, from the theory of parametric quadratic programming, see Van de Panne [1975], the optimal value function is a piecewise quadratic function of the RHS coefficients r_A, r_B, r_{AB} of the dual problem.

In each region where $z^*(r)$ is a quadratic form, the Hessian of $z^*(r)$ exists and is a symmetric positive semi-definite matrix, because

$z^*(r)$ is convex, whereby the matrix M given in (iii) of the theorem is symmetric and negative semi-definite. The formula given in (iii) follows from Taylor's theorem and the fact that $z^*(r)$ is piecewise-quadratic and therefore has no n-order terms beyond n=2. [X]

The proof that M is negative definite (and hence $z^*(r)$ is strictly convex) when $\bar{K} > 0$ is deferred until the next section, and follows as part of the proof of Theorem 4.

One important point regarding Theorem 3 deserves further elaboration. The theory of parametric quadratic programming, see e.g. Van de Panne [1975], asserts that $z^*(r_A, r_B, r_{AB})$ is a convex piecewise-quadratic function. Furthermore, there are a finite number J of closed polyhedral regions $S^1, \ldots, S^J$ in the space R^3 of values of $r = (r_A, r_B, r_{AB})$ for which the function $z^*(\cdot)$ is a quadratic form in each region S^j, $j=1,\ldots,J$. Each of these regions can be presumed to be 3-dimensional. The set of boundary points $B \subset R^3$ at which $z^*(\cdot)$ has no Hessian, i.e., no second-partial derivatives is given by $B = \bigcup_{j=1}^{J} \partial S^j$, and is a set of Lebesgue measure zero in R^3. Thus the formula of assertion (iii) of Theorem 3 is valid for all $r = (r_A, r_B, r_{AB})$ except for those $r \in B$; and B has measure zero.

3. Directional Properties of the Optimal Capacity Function

In this section, we examine the sensitivity of the optimal capacity levels $\bar{K}_A$, $\bar{K}_B$, and $\bar{K}_{AB}$ to changes in capacity costs r_A, r_B, r_{AB}. Under assumptions A1-A5, Theorem 1 ensures that $\bar{K}_A$, $\bar{K}_B$, and $\bar{K}_{AB}$ are functions of r_A, r_B, r_{AB}, and so can be written as $\bar{K}_A(r_A, r_B, r_{AB})$, $\bar{K}_B(r_A, r_B, r_{AB})$, and $\bar{K}_{AB}(r_A, r_B, r_{AB})$, and the vector $\bar{K} = (\bar{K}_A, \bar{K}_B, \bar{K}_{AB})$

can be written as $\bar{K}(r_A, r_B, r_{AB})$. We will refer to $\bar{K} = (\bar{K}_A, \bar{K}_B, \bar{K}_{AB})$ as the optimal capacity function, but will usually omit the parenthesized form "(r_A, r_B, r_{AB})" for notational convenience. Our interest lies in determining properties of the optimal capacity function relative to changes in (r_A, r_B, r_{AB}). In particular, the matrix of optimal capacity/cost partial derivatives,

$$\begin{bmatrix} \frac{\partial \bar{K}_A}{\partial r_A} & \frac{\partial \bar{K}_B}{\partial r_A} & \frac{\partial \bar{K}_{AB}}{\partial r_A} \\ \frac{\partial \bar{K}_A}{\partial r_B} & \frac{\partial \bar{K}_B}{\partial r_B} & \frac{\partial \bar{K}_{AB}}{\partial r_B} \\ \frac{\partial \bar{K}_A}{\partial r_{AB}} & \frac{\partial \bar{K}_B}{\partial r_{AB}} & \frac{\partial \bar{K}_{AB}}{\partial r_{AB}} \end{bmatrix} \qquad (2)$$

when it exists, will give valuable information regarding the direction and magnitude of changes in optimal capacity levels relative to changes in capacity costs.

Corollary 3 (Characterization of the Optimal Capacity Function). Under assumptions A1-A5, if $r = (r_A, r_B, r_{AB})$ is in the interior of a region where $z^*(r)$ is a quadratic form, and $\bar{K} = (\bar{K}_A, \bar{K}_B, \bar{K}_{AB})$ are optimal capacity values for $r = (r_A, r_B, r_{AB})$, then $\bar{K}(r+\Delta r) = \bar{K}(r) + M(\Delta r)$, for Δr sufficiently small, where M is the matrix given in (iii) of Theorem 3. In particular, M is precisely the matrix of optimal capacity/cost partial derivatives defined in (2).

PROOF: This result is an immediate consequence of Theorem 3, parts (ii) and (iii). From part (ii), $(-\bar{K}_A, -\bar{K}_B, -\bar{K}_{AB})$ is the gradient of $z^*(r)$, and the matrix M is precisely the negative of the Hessian of $z^*(r)$. Thus, for example,

$$M_{A,AB} = -\frac{\partial^2 z^*}{\partial r_A\, \partial r_{AB}} = -\frac{\partial(\partial z^*/\partial r_A)}{\partial r_{AB}} = \frac{\partial \bar{K}_A}{\partial r_{AB}}$$, and similarly,

$$M_{A,A} = \frac{\partial \bar{K}_A}{\partial r_A}, \quad M_{B,B} = \frac{\partial \bar{K}_B}{\partial r_B}, \quad M_{AB,AB} = \frac{\partial \bar{K}_{AB}}{\partial r_{AB}},$$

$$M_{A,B} = \frac{\partial \bar{K}_A}{\partial r_B}, \text{ and } M_{B,AB} = \frac{\partial \bar{K}_B}{\partial r_{AB}}.$$ Therefore, M is the matrix of optimal capacity/cost partial derivates given in (2). From Corollary 1, $\bar{K}(r)$ is piecewise linear, whereby $\bar{K}(r+\Delta r) = \bar{K}(r) + M(\Delta r)$ for Δr sufficiently small. [X]

As part of the proof of the next theorem, we will show how to compute the matrix M. Hence, corollary 3 can be viewed as constructive.

Our major result of this section is:

<u>Theorem 4 (Optimal Capacity/cost Directions and Magnitudes)</u>. Under assumptions A1-A5, if $\bar{K}(r) > 0$, then

(1) (A) $\bar{K}_A$ is strictly decreasing in r_A

(B) $\bar{K}_B$ is strictly decreasing in r_B

(AB) $\bar{K}_{AB}$ is strictly decreasing in r_{AB}

(2) (AB) $\bar{K}_A$ is strictly increasing in r_{AB}

$\bar{K}_B$ is strictly increasing in r_{AB}

(A) $\bar{K}_{AB}$ is strictly increasing in r_A

(B) $\bar{K}_{AB}$ is strictly increasing in r_B

(3) (A) $\bar{K}_B$ is strictly decreasing in r_A

(B) $\bar{K}_A$ is strictly decreasing in r_B

(4) (A) $\bar{K}_A + \bar{K}_{AB}$ is decreasing in r_{AB}

(B) $\bar{K}_B + \bar{K}_{AB}$ is decreasing in r_{AB}

(5) (A) $\bar{K}_A + \bar{K}_{AB}$ is decreasing in r_A

(B) $\bar{K}_{AB} + \bar{K}_B$ is increasing in r_A

(6) (A) $\bar{K}_B + \bar{K}_{AB}$ is decreasing in r_B

(B) $\bar{K}_{AB} + \bar{K}_A$ is increasing in r_B

Before proceeding to the proof of this theorem, we present an interpretation of this theorem and its immediate consequences.

Taken together, statements (1)-(AB) and (2)-(AB) assert that as r_{AB} increases, the firm will purchase less flexible capacity and more of each type of nonflexible capacity. Statements (1)-(A), (2)-(A), and (3)-(A) assert that as r_A increases, the firm will purchase less A-capacity, more AB-capacity and less B-capacity. Statements (1)-(A) and (2)-(A) are quite intuitive. Statement (3)-(A) follows because the substitution of flexible capacity for A-capacity induced by an increase in r_A also reduces the value, and hence the need, for B-capacity, because flexible capacity also substitutes for B-capacity. Statements (1)-(B), (2)-(B) and (3)-(B) are analogous to the above statements, for product B.

Assertions (4), (5) and (6) of the theorem indicate a "ripple" effect due to changes in capacity costs. According to statement (4), as the cost of AB-capacity is increased, the decrease in flexible capacity is larger in magnitude than the increase in either type of nonflexible capacity.

Statement (5) has a similar interpretation. For example, when the cost of nonflexible A-capacity is increased, the magnitude of the change (a decrease) in A-capacity is the largest, followed by the change in AB-capacity (an increase), and then by the change in B-capacity (a decrease). The A-capacity is the most affected, the B-capacity is the least affected, with the AB-capacity falling between the two nonflexible capacities. The "ripple" effect is A to AB to B. Statement (6) of the theorem is analogous to statement (5), for product B.

Before presenting the formal proof of Theorem 4 we first present three antecedent lemmas, preceded by an explanation of the underlying economic and mathematical concepts used in the proofs. Recall that from corollary 3, $\bar{K}(r+\Delta r) = \bar{K}(r) + M(\Delta r)$ for Δr sufficiently small (except for $r \in B$, where B is the union of the boundaries of the finitely many polyhedral convex sets s_j in the space of $r \in R^3$). Because $\bar{K}(r)$ is continuous in r under assumptions A1-A5, it suffices to prove Theorem 4 when $r \notin B$, and hence the proof of Theorem 4 becomes a proof of certain properties of the matrix M of optimal capacity/cost partial derivatives. For example, statement (1)-(A) becomes $M_{A,A} < 0$, statement (4)-(B) becomes $M_{B,A} + M_{AB,A} \leq 0$, etc. In order to prove the theorem, then, we will explictly construct the matrix M.

As a means toward computing the matrix M of optimal capacity/cost partial derivatives, we will proceed by first solving the state subproblems P(i), i=1,...,k, and then examining the parametric solution to P(i) as $\bar{K}$ changes. We then will use this analysis to construct a matrix N that will satisfy $-N(\Delta\bar{K}) = \Delta r$, where $\Delta\bar{K} = \bar{K}(r+\Delta r) - \bar{K}(r)$ for all sufficiently small Δr. The matrix M of Theorem 3 and corollary 3 is given by $M = -N^{-1}$. We then will examine the sign and magnitude of specific elements of M.

Thus our first task will be to solve the state subproblems P(i), i=1,...,k. Let $\bar{K} = (\bar{K}_A, \bar{K}_B, \bar{K}_{AB})$ be the optimal capacity values for capacity costs $r = (r_A, r_B, r_{AB})$ satisfying assumptions A1-A5. Note that the economics of subproblem P(i) implies that the marginal profit from producing an extra unit of product A is decreasing in $X_{Ai} = (Y_{Ai} + Z_{Ai})$ and reaches zero at $X_{Ai} = b_i/2a_i$. Similarly, for product B, the marginal profit is decreasing and reaches zero when at $X_{Bi} = d_i/2c_i$. The optimal solution to the subproblem P(i) will depend on the quantities $b_i/2a_i$ and $d_i/2c_i$ and their relationship to the optimal capacity values $\bar{K}_A$, $\bar{K}_B$, and $\bar{K}_{AB}$. Figure 2 plots six regions in the space of $(b_i/2a_i)$ and $(d_i/2c_i)$, bounded by the inequalities indicated in the figure. In each of the regions numbered #1-#6, the optimal solution to the subproblem P(i) is given according to Table 1, which also gives the optimal K-K-T multipliers and the algebraic description of each region. As figure 2 indicates, the regions #1-#6 given by the inequalities in Table 1 are nonoverlapping. Each of the six regions of Table 1 has a natural economic interpretation as follows:

Region 1. In this region there is enough capacity $\bar{K}_A$, $\bar{K}_B$, and $\bar{K}_{AB}$ for A production to attain $b_i/2a_i$ and B production to attain $d_i/2c_i$. Therefore, the capacity constraints in subproblem P(i) are not binding, and there is no need to allocate scarce capacity.

Region 2. In this region, there is ample A-capacity to attain zero marginal profit from product A ($\bar{\alpha}_i = 0$), but not enough B- and AB-capacity for product B to attain zero marginal profit. Thus product B uses up all of the B- and AB-capacity, which will have positive shadow values at the optimum (i.e., $\bar{\beta}_i = \bar{\gamma}_i > 0$).

Region 3. In this region, there is ample B-capacity to attain zero marginal profit of B production ($\bar{\beta}_i = 0$), but not enough A- and AB-capacity for product A to attain zero marginal value ($\bar{\alpha}_i = \bar{\gamma}_i > 0$). Thus product A uses up all of the A- and AB-capacity.

Region 4. There is insufficient A- or B-capacity to attain zero marginal value of either product's production without resorting to the flexible AB-capacity. Furthermore, the marginal profit of A-production so dominates that of B-production that all flexible capacity is devoted to A-production ($\bar{\alpha}_i = \bar{\gamma}_i > \bar{\beta}_i > 0$).

Region 5. There is insufficient A- or B-capacity to attain zero marginal value of either product's production without resorting to the flexible AB-capacity. Furthermore, the marginal value of B-production so dominates that of A-production that all flexible capacity is devoted to B-production ($\bar{\beta}_i = \bar{\gamma}_i > \bar{\alpha}_i > 0$).

Region 6. There is insufficient A- or B-capacity to attain zero marginal value of either product's production without resorting to the flexible AB-capacity. Both products share the AB-capacity and the marginal value of A-production and B-production are the same ($\bar{\alpha}_i = \bar{\beta}_i = \bar{\gamma}_i > 0$).

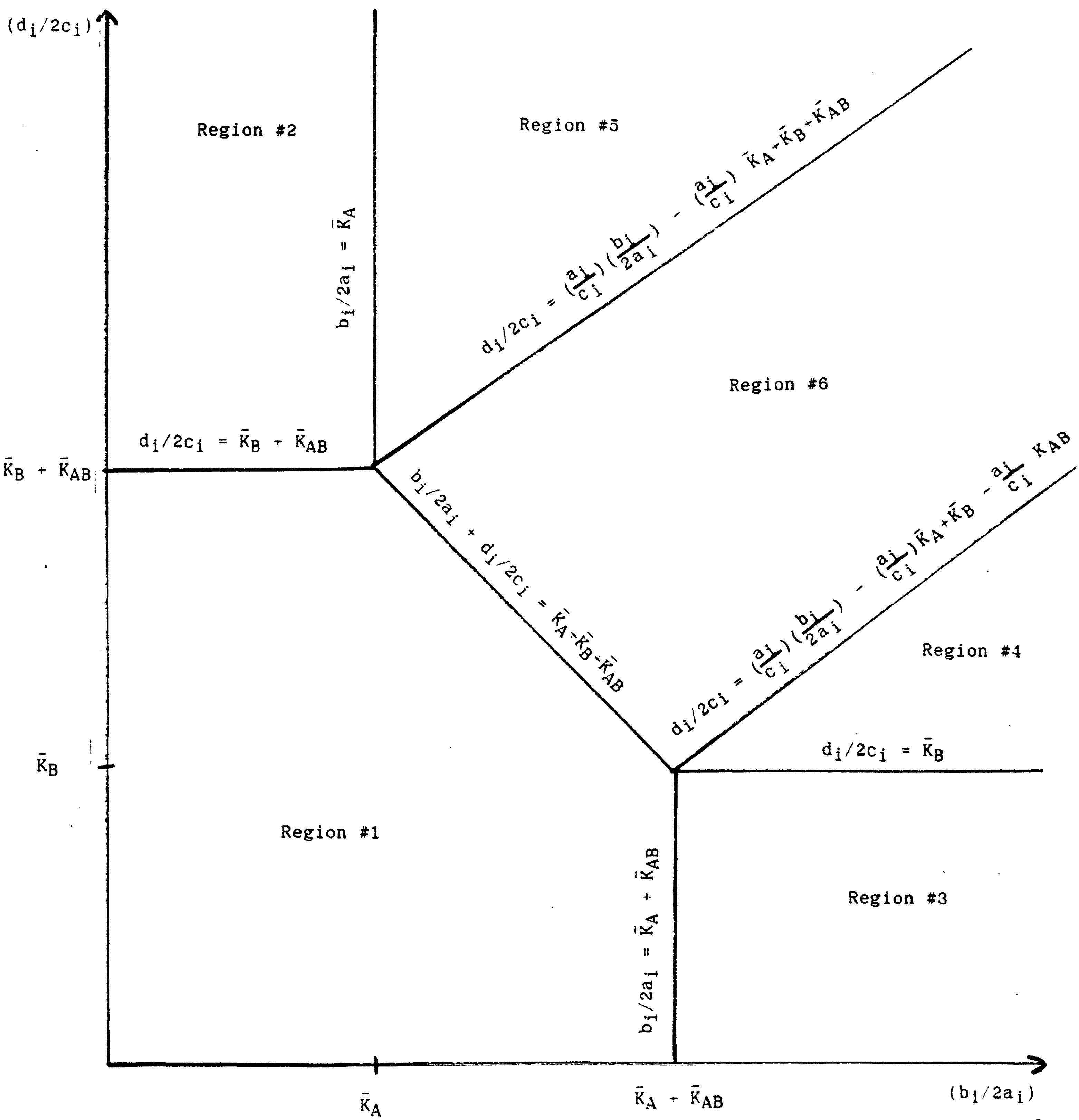

Figure 2 - The Six Regions of Lemma 2

Table 1

Optimal Subproblem Solutions for Given Capacity Levels $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$

Region #	Region Inequalities	$\bar{Y}_{Ai}$	$\bar{Z}_{Ai}$	$\bar{Y}_{Bi}$
#1	$\bar{K}_A + \bar{K}_{AB} \geq b_i/2a_i$ $\bar{K}_B + \bar{K}_{AB} \geq d_i/2c_i$ $\bar{K}_A + \bar{K}_B + \bar{K}_{AB} \geq b_i/2a_i + d_i/2c_i$	$\min\left(\frac{b_i}{2a_i}, \bar{K}_A\right)$	$\max\left(\frac{b_i}{2a_i} - \bar{K}_A, 0\right)$	$\min\left(\frac{d_i}{2c_i}, \bar{K}_B\right)$
#2	$\bar{K}_A \geq b_i/2a_i$ $\bar{K}_B + \bar{K}_{AB} \leq d_i/2c_i$	$\frac{b_i}{2a_i}$	0	$\bar{K}_B$
#3	$\bar{K}_B \geq d_i/2c_i$ $\bar{K}_A + \bar{K}_{AB} \leq b_i/2a_i$	$\bar{K}_A$	$\bar{K}_{AB}$	$\frac{d_i}{2c_i}$
#4	$\bar{K}_B \leq d_i/2c_i$ $b_i - 2a_i(\bar{K}_A + \bar{K}_{AB}) \geq d_i - 2c_i \bar{K}_B$	$\bar{K}_A$	$\bar{K}_{AB}$	$\bar{K}_B$
#5	$\bar{K}_A \leq b_i/2a_i$ $d_i - 2c_i(\bar{K}_B + \bar{K}_{AB}) \geq b_i - 2a_i \bar{K}_A$	$\bar{K}_A$	0	$\bar{K}_B$
#6	$\bar{K}_A + \bar{K}_B + \bar{K}_{AB} \leq b_i/2a_i + d_i/2c_i$ $b_i - 2a_i(\bar{K}_A + \bar{K}_{AB}) \leq d_i - 2c_i\bar{K}_B$ $d_i - 2c_i(\bar{K}_B + \bar{K}_{AB}) \leq b_i - 2a_i\bar{K}_A$	$\bar{K}_A$	$\frac{(b_i - 2a_i\bar{K}_A - d_i + 2c_i\bar{K}_B + 2c_i\bar{K}_{AB})}{(2a_i + 2c_i)}$	$\bar{K}_B$

Table 1 (continued)

Region #	$\bar{Z}_{Bi}$	$\bar{s}_i$	$\bar{u}_i$	$\bar{t}_i$	$\bar{v}_i$	$\bar{\alpha}_i$	$\bar{\beta}_i$	$\bar{\gamma}_i$
#1	$\max\left(\frac{d_i}{2c_i}-\bar{K}_B,\ 0\right)$	0	0	0	0	0	0	0
#2	$\bar{K}_{AB}$	0	0	0	0	0	$d_i - 2c_i\bar{K}_B - 2c_i\bar{K}_{AB}$	$d_i - 2c_i\bar{K}_B - 2c_i\bar{K}_{AB}$
#3	0	0	0	0	0	$b_i - 2a_i\bar{K}_A - 2a_i\bar{K}_{AB}$	0	$b_i - 2a_i\bar{K}_A - 2a_i\bar{K}_{AB}$
#4	0	0	0	0	$b_i - 2a_i\bar{K}_A - 2a_i\bar{K}_{AB} - d_i + 2c_i\bar{K}_B$	$b_i - 2a_i\bar{K}_A - 2a_i\bar{K}_{AB}$	$d_i - 2c_i\bar{K}_B$	$b_i - 2a_i\bar{K}_A - 2a_i\bar{K}_{AB}$
#5	$\bar{K}_{AB}$	0	$d_i - 2c_i\bar{K}_B - 2c_i\bar{K}_{AB} - b_i + 2a_i\bar{K}_A$	0	0	$b_i - 2a_i\bar{K}_A$	$d_i - 2c_i\bar{K}_B - 2c_i\bar{K}_{AB}$	$d_i - 2c_i\bar{K}_B - 2c_i\bar{K}_{AB}$
#6	$\frac{(d_i - 2c_i\bar{K}_B - b_i + 2a_i\bar{K}_A + 2a_i\bar{K}_{AB})}{(2a_i + 2c_i)}$	0	0	0	0	$\frac{-2a_ic_i(\bar{K}_A + \bar{K}_B + \bar{K}_{AB}) + b_ic_i + a_id_i}{a_i + c_i}$		

It is straightforward to verify that the K-K-T conditions are satisfied for each region in Table 1, and so the solutions given are optimal for subproblem P(i). Furthermore, in each region in the table, the K-K-T multipliers $\bar{s}_i$, $\bar{t}_i$, $\bar{u}_i$, $\bar{v}_i$, $\bar{\alpha}_i$, $\bar{\beta}_i$ and $\bar{\gamma}_i$ satisfy the formulas (1) of lemma 1.

With $r = (r_A, r_B, r_{AB})$ given, let us assume that $r \notin B$. Let $\Delta r = (\Delta r_A, \Delta r_B, \Delta r_{AB})$ be a vector of small capacity cost changes. Let $\Delta K = \bar{K}(r+\Delta r) - \bar{K}(r)$ be the change in the optimal capacity values induced by the change in capacity costs Δr. According to corollary 3, $\Delta\bar{K} = M(\Delta r)$, where M is the matrix of optimal capacity/cost partial derivatives. Knowing $\Delta\bar{K} = (\Delta\bar{K}_A, \Delta\bar{K}_B, \Delta\bar{K}_{AB})$ allows us to easily compute the per unit changes in the subproblem shadow prices $\bar{\alpha}_i$, $\bar{\beta}_i$, $\bar{\gamma}_i$ from Table 1. For example, if the data for state i results in $(\frac{b_i}{2a_i}, \frac{d_i}{2c_i})$ falling in region #4, then $\bar{\alpha}_i = b_i - 2a_i\bar{K}_A - 2a_i\bar{K}_{AB}$, and the per unit change in $\bar{\alpha}_i$ relative to a change in $\bar{K}_{AB}$ is $-2a_i$. These per unit changes in $\bar{\alpha}_i$, $\bar{\beta}_i$, $\bar{\gamma}_i$ relative to changes in $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$ (derived from Table 1) are shown in Table 2.

For each subproblem P(i), let the matrix N^i denote the negative per unit changes in $\bar{\alpha}_i$, $\bar{\beta}_i$, and $\bar{\gamma}_i$ relative to a change in $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$. For example, the element $n^i_{B,AB}$ of N^i is the negative of the per unit change $\bar{\beta}_i$ relative to a change in $\bar{K}_{AB}$. This denotation appears in the bottom row of Table 2. Define the matrix N by

$$N = \sum_{i=1}^{k} N^i \quad . \qquad (3)$$

By definition, $n_{B,AB}$, for example, represents the sum (over all states i) of the negative per unit change in $\bar{\beta}_i$ relative to a change in $\bar{K}_{AB}$,

Table 2

Per Unit Changes in $\bar{\alpha}_i$, $\bar{\beta}_i$, $\bar{\gamma}_i$ relative to changes $\Delta\bar{K}_A$, $\Delta\bar{K}_B$, $\Delta\bar{K}_B$

Region	Change in $\bar{\alpha}_i$ per unit change in:			Change in $\bar{\beta}_i$ per unit change in:			Change in $\bar{\gamma}_i$ per unit change in:		
	$\bar{K}_A$	$\bar{K}_B$	$\bar{K}_{AB}$	$\bar{K}_A$	$\bar{K}_B$	$\bar{K}_{AB}$	$\bar{K}_A$	$\bar{K}_B$	$\bar{K}_{AB}$
#1	0	0	0	0	0	0	0	0	0
#2	0	0	0	0	$-2c_i$	$-2c_i$	0	$-2c_i$	$-2c_i$
#3	$-2a_i$	0	$-2a_i$	0	0	0	$-2a_i$	0	$-2a_i$
#4	$-2a_i$	0	$-2a_i$	0	$-2c_i$	0	$-2a_i$	0	$-2a_i$
#5	$-2a_i$	0	0	0	$-2c_i$	$-2c_i$	0	$-2c_i$	$-2c_i$
#6	$\frac{-2a_ic_i}{a_i+c_i}$	$\frac{-2a_ic_i}{a_i+c_i}$	$\frac{-2a_ic_i}{a_i+c_i}$	$\frac{-2a_ic_i}{a_i+c_i}$	$\frac{-2a_ic_i}{a_i+c_i}$	$\frac{-2a_ic_i}{a_i+c_i}$	$\frac{-2a_ic_i}{a_i+c_i}$	$\frac{-2a_ic_i}{a_i+c_i}$	$\frac{-2a_ic_i}{a_i+c_i}$
Denotation	$-n^i_{A,A}$	$-n^i_{A,B}$	$-n^i_{A,AB}$	$-n^i_{B,A}$	$-n^i_{B,B}$	$-n^i_{B,AB}$	$-n^i_{AB,A}$	$-n^i_{AB,B}$	$-n^i_{AB,AB}$

i.e., $n_{B,AB}$ is the negative change in the $\sum_{i=1}^{k} \bar{\beta}_i$ relative to a change in $\bar{K}_{AB}$. If $\bar{K}_B > 0$, then $\sum_{i=1}^{k} \bar{\beta}_i = r_B$, and so $n_{B,AB}$ is the negative per unit change in r_B relative to a change in $\bar{K}_{AB}$. Using similar logic for the other components of N, we have:

Lemma 2. Under assumptions A1-A5, if $r = (r_A, r_B, r_{AB}) \notin B$, and $\bar{K}(r) > 0$, then the matrix N defined by Table 1, Table 2, and (3), satisfies

$$N(\Delta\bar{K}) = -(\Delta r) \quad . \qquad [X]$$

If the matrix N is invertible, then $\Delta\bar{K} = (-N^{-1})(\Delta r)$, whereby we must have $M = -N^{-1}$. Thus, our next task is to prove

Lemma 3. Under assumptions A1-A5, the matrix N defined by Table 1, Table 2, and (3) has a positive determinant. Furthermore, the matrix N can be written as

$$N = \begin{bmatrix} \bar{e}+\bar{b} & \bar{e} & \bar{e}+\bar{a} \\ \bar{e} & \bar{e}+\bar{d} & \bar{e}+\bar{c} \\ \bar{e}+\bar{a} & \bar{e}+\bar{c} & \bar{e}+\bar{a}+\bar{c} \end{bmatrix} , \qquad (4)$$

where

$$\bar{a} = \sum_{i \in R_3 \cup R_4} 2a_i \quad ,$$

$$\bar{b} = \sum_{i \in R_3 \cup R_4 \cup R_5} 2a_i$$

$$\bar{c} = \sum_{R_2 \cup R_5} 2c_i \quad ,$$

$$\bar{d} = \sum_{i \in R_2 \cup R_4 \cup R_5} 2c_i \quad ,$$

$$\bar{e} = \sum_{i \in R_6} \frac{2a_i c_i}{a_i + c_i} \quad ,$$

and R_j denotes the set of state i that are in region j as defined in Table 1 (or Figure 2), j=1,...,6.

PROOF: See Appendix. [X]

By Cramer's rule, we obtain:

Lemma 4. Under assumptions A1-A5, if $r = (r_A, r_B, r_{AB}) \notin B$, and $\bar{K}(r) > 0$, then

$$M = \frac{1}{\det(N)} \begin{bmatrix} n_{B,AB}\, n_{B,AB} - n_{B,B}\, n_{AB,AB} & n_{A,B}\, n_{AB,AB} - n_{A,AB}\, n_{B,AB} & n_{B,B}\, n_{A,AB} - n_{A,B}\, n_{B,AB} \\ n_{A,B}\, n_{AB,AB} - n_{A,AB}\, n_{B,AB} & n_{A,AB}\, n_{A,AB} - n_{A,A}\, n_{AB,AB} & n_{A,A}\, n_{B,AB} - n_{A,B}\, n_{A,AB} \\ n_{B,B}\, n_{A,AB} - n_{A,B}\, n_{B,AB} & n_{A,A}\, n_{B,AB} - n_{A,B}\, n_{A,AB} & n_{A,B}\, n_{A,B} - n_{A,A}\, n_{B,B} \end{bmatrix}$$

[X]

With the formula for M given in lemma 3, we are in position to prove Theorem 4, which describes how optimal capacity levels change as a function of the capacity costs.

Proof of Theorem 4: Given the formula for the matrix M from lemma 4 and from equation (4), the proof of Theorem 4 is accomplished by verifying certain inequality relationships among the elements of the matrix M. The details of this procedure are in the Appendix [X].

Theorem 4 is conditioned on the supposition that all optimal capacity values are positive. When one of the optimal capacity values, say $\bar{K}_A$, is zero, we obviously can no longer assert that $\bar{K}_A$ is strictly decreasing in r_A. Furthermore, the computation of the matrix M cannot be accomplished as in the proof of Theorem 4, because we may not have $\sum_{i=1}^{k} \bar{\alpha}_i = r_A$, i.e., $\bar{m}$ could be positive. However, using logic similar to that in Theorem 4, we can prove:

Corollary 4: Under assumptions A1-A5, the six assertions of Theorem 4 remain valid, without the strict monotonicity in assertions (1)-(3).

4. Example with Calculation of the Value of Flexibility

Consider a firm that produces two basic product lines, A and B, and faces uncertainty about the future demand for each product line. If this firm must make its investment decisions on flexible and on nonflexible capacity levels before the demand uncertainty is resolved, then our model provides a useful tool for analysis of the investment decision problem. One instance of such a setting occurs in the auto industry. In their engine plants, automobile manufacturers face the choice of building nonflexible machining lines that can only produce one size engine block or flexible lines that can switch, for example, between four and six cylinder engines. Since engine plant construction lead times and useful lives are long relative to the frequency of oil price movements (which affect consumer demand for cars with small or large engines), automobile manufacturers may have an incentive to build some flexible engine line capacity that can be used for either large or small engines.

The following example illustrates the calculations that such an automobile manufacturer could perform to analyze its product-flexible manufacturing system investment decision. We will also illustrate, for this example, a calculation of the value of flexibility as a function of the per unit cost of flexible capacity. The value of flexibility as a function of r_{AB} is measured by $z^*(r_A, r_B, r_{AB}) - z^*(r_A, r_B, \infty)$, the additional profits obtained when flexible capacity is available at a cost of r_{AB} per unit. The numerical data presented do not reflect actual industry conditions. They are only used for purposes of illustration.

Suppose there are four possible future states of the world, depending on whether oil prices are high or low, and whether the U.S.

gross national product growth is high or low. Further, suppose demand for products A and B (small and large engines) is linear in the prices of A and B, respectively, and that the state of the world only affects the vertical intercept of the products' demand functions. Thus, in the notation of the previous sections, the random variables are $b_i=p_i e_i$ and $d_i=p_i g_i$, for i=1,2,3,4. For simplicity, we assume the product demand slopes are unity, i.e., $f_i=h_i=1$ for all i (so that $a_i=c_i=p_i$). The data for the example are as follows:

State i	Oil Prices	GNP Growth	p_i	Small engine (Product A) demand intercept, e_i	Large engine (Product B) demand intercept, g_i	Large and Small Engine demand slope f_i, h_i
1	High	High	.10	18	10	1
2	High	Low	.30	16	6	1
3	Low	High	.40	8	18	1
4	Low	Low	.20	4	16	1

Further, we assume the per unit capacity costs are $r_A = 2.6$, $r_B = 2.2$, $r_{AB} = 3.8$.

The table below shows the data for each state expressed in terms of the quantities a_i, b_i, c_i, d_i.

State i	a_i	b_i	c_i	d_i
1	.10	1.8	.10	1.0
2	.30	4.8	.30	1.8
3	.40	3.2	.40	7.2
4	.20	.8	.20	3.2

Clearly, the assumptions A1 A5 are satisfied, and so this PFMS problem exhibits a unique solution. This optimal solution is

$\bar{K}_A = 3$, $\bar{K}_B = 4$, $\bar{K}_{AB} = 3$. Since non flexibile capacity costs less than flexible capacity, the firm optimally chooses to hold some nonflexible capacity. However, the uncertainty in the demands for the two products make purchase of 3 units of the more expensive flexible capacity efficient. The dual prices α_i, β_i, and γ_i, as well as the region (from Figure 2 in the previous section) for each state are given below.

State	α_i	β_i	γ_i	Region #
1	.6	.2	.6	4
2	1.2	0	1.2	3
3	.8	1.6	1.6	5
4	0	.4	.4	2

Straightforward calculation shows that $z^*(2.6, 2.2, 3.8) = 52.5$. In order to compute the value of flexibility, we compute $z^*(2.6, 2.2, +\infty) = 50\ 1/12$ by solving the PFMS problem will $r_{AB} = +\infty$. Therefore, the value of flexibility (when $r_{AB} = 3.8$) is $52.5 - 50\ 1/12 = 2\ 5/12$.

In order to obtain N, the matrix of negative per unit changes in the capacity shadow prices relative to the capacity levels, we use Table 2 to first calculate N^i for each state i. Following this procedure, we obtain

$$N^1 = \begin{bmatrix} .2 & 0 & .2 \\ 0 & .2 & 0 \\ .2 & 0 & .2 \end{bmatrix},\ N^2 = \begin{bmatrix} .6 & 0 & .6 \\ 0 & 0 & 0 \\ .6 & 0 & .6 \end{bmatrix},\ N^3 = \begin{bmatrix} .8 & 0 & 0 \\ 0 & .8 & .8 \\ 0 & .8 & .8 \end{bmatrix},$$

$$N^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .4 & .4 \\ 0 & .4 & .4 \end{bmatrix}, \text{ and } N = \begin{bmatrix} 1.6 & 0 & .8 \\ 0 & 1.4 & 1.2 \\ .8 & 1.2 & 2 \end{bmatrix}$$

and $M = -N^{-1} = \begin{bmatrix} -17/16 & -3/4 & 7/8 \\ -3/4 & -2 & 1.5 \\ 7/8 & 1.5 & -1.75 \end{bmatrix}$.

Thus,

$$z^*(2.6+\Delta r_A,\ 2.2+\Delta r_B,\ 3.8+\Delta r_{AB}) = 52.5 - 3(\Delta r_A) - 4(\Delta r_B) - 3(\Delta r_{AB})$$

$$+ 1/2 \begin{pmatrix} \Delta r_A \\ \Delta r_B \\ \Delta r_{AB} \end{pmatrix}^T \begin{bmatrix} -17/16 & -3/4 & 7/8 \\ -3/4 & -2 & 1.5 \\ 7/8 & 1.5 & -1.75 \end{bmatrix} \begin{pmatrix} \Delta r_A \\ \Delta r_B \\ \Delta r_{AB} \end{pmatrix}.$$

In particular, for example

$$\frac{\partial z^*}{\partial r_{AB}} = -3 \quad \text{and} \quad \frac{\partial \bar{K}_A}{\partial r_{AB}} = \frac{7}{8} .$$

That is, with an incremental increase in the cost of flexible capacity, optimal profits decrease at the rate of three units, and optimal dedicated A-capacity increases at the rate of .875 units.

5. Discussion

Our model provides the beginnings of a conceptual framework for evaluating investments in product-flexible manufacturing systems. To ease the analysis of the model, we have assumed that the firm produces only two different products and can invest in only three different technologies. An alternate formulation, due to Kulatilaka [1985], allows M different types of nonflexible technologies, each of which may be optimal in certain states; and analyzes the optimal use over an infinite horizon of a flexible technology whose "mode" can be switched each period (at a cost) so as to emulate any of the M nonflexible technologies. The modelling cost of this richer formulation is that Kulatilaka makes a very specific assumption on the distribution of the sequence of random variables relevant to the optimal single-period technology choice. (The random process is a discrete version of mean-reverting Weiner process.) Kulatilaka is able to calculate optimal policies for specific numerical examples.

In contrast, our model, which we restrict to only two products, assumes that the firm may invest in a portfolio of flexible and nonflexible technologies and that the switching costs are either zero (for the flexible technology) or infinite (for the nonflexible technologies.) Our model formulation PFM1 can easily accommodate multiple time periods by adding a time index to XA_i, XB_i, p_i, e_i, f_i, g_i, h_i, summing the profits over the time periods in the objective function, and adding a discount factor. The analysis remains the same, assuming the firm does not carry inventories from period to period. If we permit inventories, then the analysis of the model becomes significantly more difficult. Our intuition is that the presence of inventories would reduce the value of flexible capacity to the firm since holding inventories provides flexibility for firms in meeting demand. We think that an extension of the model, exploring this conjecture, warrants further analysis.

Another extension of our model would be to assume that the per unit variable cost of production differs, depending on whether the unit is produced with the flexible or nonflexible capacity. Many analysts claim that automated flexible manufacturing systems have lower per unit variable costs than do their less flexible substitutes. When this is the case, the firm will obviously buy more flexible and less nonflexible capacity than would be the case for the formulation analyzed here. Suppose, in addition, we generalize the form of the acquisition cost to include a fixed component if any positive level of a certain technology is chosen. For example, assume that investing in $K_{AB} > 0$ units of flexible capacity costs $R_{AB} + r_{AB} \cdot K_{AB}$, where $R_{AB} > 0$. Similarly, assume nonflexible capacity investment costs are $R_A + r_A K_A$ and $R_B + r_B K_B$. In this case, if we look at the optimal levels of K_A, K_B, and K_{AB} as we increase the difference between the variable cost of producing with the flexible versus the nonflexible technology or as we increase R_A, R_B, and R_{AB}, then at some point the optimal levels of K_A and K_B will become zero and only flexible capacity will be purchased. Thus, a more realistic production cost and acquisition cost structure can yield a solution where the option of investing in nonflexible capacity is abandoned completely in favor of investment solely in flexible capacity. We think that an exploration of this extension merits further work.

A final direction for further exploration of this model is to look at how competition affects firms' optimal investment in flexible capacity. In one two-firm extension of our model, suppose that firm 1 is a monopolist in product markets A and B. If firm 2 is viewed as threatening to enter market A (only) then firm 1 may want to hold some excess A-capacity to deter entry. (See, e.g., Spence [1977], Dixit [1979], Bulow et al. [1985].) The question of interest is whether firm 1 can deter entry by holding product-flexible (K_{AB}) capacity. Holding K_{AB} to deter entry is less costly than holding only K_A to deter entry because unused K_{AB} capacity has an alternate use. However, because of this alternate use, holding K_{AB} may not be a credible threat to potential entrants.

The crucial issue here relates to the conflicting effects of commitment and flexibility. To deter entry, an established firm must appear to the entrant to be committed to behavior that will be detrimental to a new entrant. Such a commitment can be made credible, for example, by having already invested in capacity so that marginal costs of production are low (as in Dixit [1980]). Alternatively, a firm may demonstrate commitment by investing early in a new market (as in Spence [1979], and Fudenberg and Tirole [1983]). Holding product-flexible capacity would appear to reduce the credibility of commitment to high post-entry production. This is because, once entry has occurred, the flexible capacity might have a higher marginal product if deployed elsewhere. Therefore a multimarket monopolist who is threatened by potential entrants may not wish to hold flexible capacity because it detracts from the credibility of commitment to defend the threatened market.

Countering this effect, however, a multimarket monopolist will prefer to hold some flexible capacity if demand is uncertain (as in the previous sections) or if entry is uncertain. (Entry may be uncertain because the potential entrant's costs are not observable by the incumbent as in Saloner [1983], Kreps and Wilson [1982], and Milgrom and Roberts [1982a].) Further analysis is required to resolve how flexible technology affects equilibrium capacity investment when these two conflicting forces are present.

Similar questions revolve around how product-flexible capacity might be used by potential entrants. In particular, purchasing product-flexible capacity for entering a new market may be a less risky way to enter a market controlled by an incumbent with unknown costs. That is, low exit costs may reduce the costs of entry. (See Eaton and Lipsey [1980] for additional perspective on this issue.) However, an incumbent might play tougher against an entrant who is known to have product-flexible capacity that can be used in other markets. (This is especially likely if the incumbent has an opportunity for reputation formation as in Kreps and Wilson [1982] or Milgrom and Roberts [1982b]). Obviously, there is much work to be done to determine how the existence of product-flexible capacity affects competition and entry in this multi-market oligopoly entry game.

Appendix

PROOF of Theorem 1: The PFMS problem is always feasible, since $X_{Ai}=X_{Bi}=0$, $i=1,\ldots,k$, $K_A=K_B=K_{AB}=0$ is feasible. By assumption A2, the PFMS problem is a convex quadratic program, and by assumptions A2 and A3 it is bounded from above. It thus attains its optimum. Because the objective function is strictly convex in X_{Ai} and X_{Bi}, $i=1,\ldots,k$, $\bar{X}_{Ai}$, $\bar{X}_{Bi}$ are uniquely determined for each i.

It remains to show that $\bar{K}_A$, $\bar{K}_B$, $\bar{K}_{AB}$ are uniquely determined. Assuming the contrary, suppose K_A^1, K_B^1, K_{AB}^1 and K_A^2, K_B^2, K_{AB}^2 are alternative optimal values of K_A, K_B, and K_{AB}, and $(K_A^1, K_B^1, K_{AB}^1) \neq (K_A^2, K_B^2, K_{AB}^2)$. Then $r_A K_A^1 + r_B K_B^1 + r_{AB} K_{AB}^1 = r_A K_A^2 + r_B K_B^2 + r_{AB} K_{AB}^2$. We have two cases:

Case 1. $K_{AB}^1 = K_{AB}^2$. Then $K_A^1 \neq K_A^2$ and $K_B^1 \neq K_B^2$, and we can assume that $K_A^1 > K_A^2 \geq 0$, and hence $K_B^2 > K_B^1 \geq 0$. Let $(\bar{\delta}_i, \bar{\theta}_i, \bar{\lambda}_i, \bar{q}_i, \bar{r}_i, i=1,\ldots,k, \bar{m}, \bar{n}, \bar{p})$ be any optimal K-K-T multipliers. Then $\bar{X}_{Ai} \leq K_A^2 < K_A^1$, whereby $\bar{\delta}_i = 0$, $i=1,\ldots,k$. Similarly $\bar{X}_{Bi} \leq K_B^1 < K_B^2$, and so $\bar{\theta}_i = 0$, $i=1,\ldots,k$. Also, since $K_A^1 > 0$, and $K_B^2 > 0$, $\bar{m} = 0$ and $\bar{n} = 0$. Thus $r_A = \sum_{i=1}^{k} \bar{\lambda}_i$, and $r_B = \sum_{i=1}^{k} \bar{\lambda}_i$, and

$$r_{AB} = \sum_{i=1}^{k} \bar{\lambda}_i + \bar{p}.$$

If $\bar{p} = 0$, then $r_{AB} = r_A$, violating assumption A4. Thus $\bar{p} > 0$. But then $K_{AB}^1 = K_{AB}^2 = 0$. For each index $i=1,\ldots,k$, $\bar{X}_{Ai} \leq K_A^2 + K_{AB}^2 = K_A^2$ and $\bar{X}_{Bi} \leq K_B^1 + K_{AB}^1 = K_B^1$, and so $\bar{X}_{Ai} + \bar{X}_{Bi} \leq K_A^2 + K_B^1 < K_A^1 + K_B^1 = K_A^1 + K_B^1 + K_{AB}^1$, and so $\bar{\lambda}_i = 0$, $i=1,\ldots,k$. Thus $r_A = 0$, a contradiction.

Case 2: $K^1_{AB} \neq K^2_{AB}$. Without loss of generality, we can assume that $K^1_{AB} > K^2_{AB} \geq 0$, and so $\bar{p} = 0$. Suppose that $K^1_A + K^1_{AB} \neq K^2_A + K^2_{AB}$, in addition. Then $\bar{\delta}_i = 0$, i=1,...,k, whereby $r_{AB} = \sum_{i=1}^{k} (\bar{\delta}_i + \bar{\theta}_i + \bar{\lambda}_i) = \sum_{i=1}^{k} (\bar{\theta}_i + \bar{\lambda}_i) = r_B - \bar{n}$. This implies that $r_{AB} \leq r_B$, violating assumption A4. Therefore $K^1_A + K^1_{AB} = K^2_A + K^2_{AB}$, and by an identical argument, it must be true that $K^1_B + K^1_{AB} = K^2_B + K^2_{AB}$. Therefore, $K^2_{AB} - K^1_{AB} = K^1_B - K^2_B = K^1_A - K^2_A$. However, it is also true that $r_{AB} (K^2_{AB} - K^1_{AB}) = r_A (K^1_A - K^2_A) + r_B (K^1_B - K^2_B)$. This forces $r_{AB} = r_A + r_B$, which violates assumption A5, proving the theorem. [X]

PROOF of Lemma 3: We note the following inequalities as a consequence of Table 2:

$$n^i_{AA} \geq n^i_{A,AB} \geq n^i_{A,B} \geq 0$$
$$n^i_{BB} \geq n^i_{B,AB} \geq n^i_{B,A} \geq 0$$
$$n^i_{AB,AB} \geq n^i_{A,AB} \geq 0$$
$$n^i_{AB,AB} \geq n^i_{B,AB} \geq 0$$
$$n^i_{A,B} = n^i_{B,A}$$
$$n^i_{AB,A} = n^i_{A,AB}$$
$$n^i_{AB,B} = n^i_{B,AB} \qquad , \ i=1,\ldots,k$$

which imply:

$$n_{AA} \geq n_{A,AB} \geq n_{A,B} \geq 0$$
$$n_{B,B} \geq n_{B,AB} \geq n_{B,A} \geq 0$$
$$n_{AB,AB} \geq n_{A,AB} \geq 0$$
$$n_{AB,AB} \geq n_{B,AB} \geq 0$$
$$n_{A,B} = n_{B,A}$$
$$n_{AB,A} = n_{A,AB}$$
$$n_{AB,B} = n_{B,AB}$$

Note, in particular, that N is symmetric and nonnegative. Now let R_j, $j=1,\ldots,6$, denote the set of states i that are in region j as defined in Table 1 (or Figure 2).

Define the following numbers:

$$\bar{a} = \sum_{i \in R_3 \cup R_4} 2a_i \quad ,$$

$$\bar{b} = \sum_{i \in R_3 \cup R_4 \cup R_5} 2a_i \quad ,$$

$$\bar{c} = \sum_{R_2 \cup R_5} 2c_i \quad ,$$

$$\bar{d} = \sum_{i \in R_2 \cup R_4 \cup R_5} 2c_i$$

$$\bar{e} = \sum_{i \in R_6} \frac{2a_i c_i}{a_i + c_i} \quad .$$

Then the matrix N can be written in the form:

$$N = \begin{bmatrix} \bar{e}+\bar{b} & \bar{e} & \bar{e}+\bar{a} \\ \bar{e} & \bar{e}+\bar{d} & \bar{e}+\bar{c} \\ \bar{e}+\bar{a} & \bar{e}+\bar{c} & \bar{e}+\bar{a}+\bar{c} \end{bmatrix} \qquad (4)$$

The determinant of N is:

$$\det(N) = \bar{a}\bar{d}(\bar{b}-\bar{a}) + \bar{b}\bar{c}(\bar{d}-\bar{c}) + \bar{e}\,[(\bar{b}-\bar{a})(\bar{d}-\bar{c}) + \bar{a}(\bar{b}-\bar{a}) + \bar{c}(\bar{d}-\bar{c}) + \bar{a}\bar{c}].$$

Because $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ are all nonnegative, and $\bar{b} \geq \bar{a}$, and $\bar{d} \geq \bar{c}$, all of the above terms are nonnegative, whereby $\det(N) \geq 0$. It remains to show that some term is strictly positive.

Assuming the contrary, we will derive a contradiction. Therefore we assume that $\bar{a}\bar{d}(\bar{b}-\bar{a}) = 0$ and $\bar{b}\bar{c}(\bar{d}-\bar{c}) = 0$. Then one of the following nine cases must be true:

Case 1. $\bar{a} = 0$, $\bar{b} = 0$. Thus all states must be in regions 1, 2, or 6. But then, by Table 1, $\bar{\beta}_i = \bar{\gamma}_i$ for all i, whereby $r_B = r_{AB}$, contradicting assumption A4.

Case 2. $\bar{a} = 0$, $\bar{c} = 0$. Thus all states must be in regions 1 or 6. By Table 1, $\bar{\beta}_i = \bar{\gamma}_i$ for all i, whereby $r_B = r_{AB}$, contradicting assumption A4.

Case 3. $\bar{a} = 0$, $\bar{d} = \bar{c}$. Thus all states must be in regions 1, 2, 5, or 6. By Table 1, $\bar{\beta}_i = \bar{\gamma}_i$ for all i, whereby $r_B = r_{AB}$, contradicting assumption A4.

Case 4. $\bar{d} = 0$, $\bar{b} = 0$. Thus all states must be in regions 1 or 6. By Table 1, $\bar{\alpha}_i = \bar{\gamma}_i$, for all i, whereby $r_A = r_{AB}$, contradicting assumption A4.

Case 5. $\bar{d} = 0$, $\bar{c} = 0$. Thus all states must be in regions 1, 3, or 6. By Table 1, $\bar{\alpha}_i = \bar{\gamma}_i$, for all i, whereby $r_A = r_{AB}$, contradicting assumption A4.

Case 6. $\bar{d} = 0$, $\bar{d} = \bar{c}$. This is identical to case 5.

Case 7. $\bar{b} = \bar{a}$, $\bar{b} = 0$. This is identical to case 1.

Case 8. $\bar{b} = \bar{a}$, $\bar{c} = 0$. Thus all states must be regions 1, 3, 4, or 6. By Table 1, $\bar{\alpha}_i = \bar{\gamma}_i$, for all i, whereby $r_A = r_{AB}$, contradicting assumption A4.

Case 9. $\bar{b} = \bar{a}$, $\bar{d} = \bar{c}$. Thus all states must be in regions 1, 2, 3, or 6. If $\bar{e} = 0$, then all states are in regions 1, 2, or 3, and $\bar{\alpha}_i + \bar{\beta}_i = \bar{\gamma}_i$, for all i. This implies that $r_A + r_B = r_{AB}$, contradicting assumption A5. Thus, $\bar{e} > 0$, and $\det(N) = \bar{e}\bar{a}\bar{c}$. Thus $\bar{a} = 0$ or $\bar{c} = 0$. If $\bar{a} = 0$, then this implies all states are in regions 1, 2, 5, or 6, and so $r_B = r_{AB}$, a contradiction. If $\bar{c} = 0$, this implies all states are in regions 1, 3, 4, or 6, and so $r_A = r_{AB}$, a contradiction.

Thus $\det(N) > 0$. [X]

Proof of Theorem 4: Without loss of generality, we can assume $\det(N) = 1$. We prove each of the six assertions separately.

(1) We must show that $m_{AA} < 0$, $m_{BB} < 0$, and $m_{AB,AB} < 0$. Because $n_{BB} \geq n_{B,AB}$ and $n_{AB,AB} \geq n_{B,AB}$, $m_{A,A} = (n_{B,AB}\, n_{B,AB} - n_{B,B}\, n_{AB,AB}) \leq 0$. If $m_{A,A} = 0$, then either (i) $n_{B,B} = 0$, (ii) $n_{AB,AB} = 0$, or (iii) $n_{B,B} = n_{B,AB} = n_{AB,AB}$. If (i), then all states are in regions 1 or 3, implying $r_A = r_{AB}$. If (ii), then all states are in regions 1, implying $r_A = r_{AB} = 0$. If (iii), then all states must be in regions 1, 2, 5, or 6, implying $r_B = r_{AB}$. Since each case results in a contradiction of assumption A4, $m_{A,A} < 0$. Similar logic can be used to deduce that $m_{B,B} < 0$ and $m_{AB,AB} < 0$.

(2) We must show that $m_{A,AB} > 0$, $m_{B,AB} > 0$, $m_{AB,A} > 0$, and $m_{AB,B} > 0$. Because $n_{B,B} \geq n_{B,AB}$ and $n_{A,AB} \geq n_{A,B}$, $m_{A,AB} = (n_{B,B}\, n_{A,AB} - n_{A,B}\, n_{B,AB}) \geq 0$. If $m_{A,AB} = 0$, then either $n_{B,B} = 0$, $n_{A,AB} = 0$, or $n_{B,B} = n_{B,AB}$ <u>and</u> $n_{A,AB} = n_{A,B}$. The first case implies that $r_A = r_{AB}$, the second implies thar $r_B = r_{AB}$, and the third implies that $r_B = r_{AB}$.

Thus we obtain a contradiction to assumption A4, and $m_{A,AB} > 0$. Similar logic can be used to show that $m_{B,AB} > 0$. Furthermore, since M is symmetric, $m_{A,AB} = m_{AB,A}$ and $m_{B,AB} = m_{AB,B}$.

(3) We must show that $m_{A,B} < 0$ and $m_{B,A} < 0$. $m_{A,B} = (n_{A,B}\, n_{AB,AB} - n_{A,AB}\, n_{B,AB}) = (\bar{e}(\bar{e}+\bar{a}+\bar{c}) - (\bar{e}+\bar{a})(\bar{e}+\bar{c})) = (-\bar{a}\bar{c}) \leq 0$, because $\bar{a} \geq 0$ and $\bar{c} \geq 0$, by (4). Furthermore, if $a = \bar{}0$, then all states must be in regions 1, 2, 5 or 6, whereby $r_B = r_{AB}$, contradicting assumption A4. Similarly, $\bar{c} > 0$, otherwise $r_A = r_{AB}$, which again is a contradiction. Thus $-\bar{a}\bar{c} < 0$, and so $m_{A,B} < 0$.

(4) We must show that $m_{A,AB} + m_{AB,AB} \le 0$ and $m_{B,AB} + m_{AB,AB} \le 0$. Note that $+ m_{AB,AB} + m_{A,AB} = (- n_{A,A} n_{B,B} + n_{A,B} n_{A,B} + n_{B,B} n_{A,AB} - n_{A,B} n_{B,AB})$. But $-n_{A,A} n_{B,B} + n_{A,B} n_{A,B} + n_{B,B} n_{A,AB} - n_{A,B} n_{B,AB} = n_{B,B}(- n_{A,A} + n_{A,AB}) - n_{A,B}(n_{B,AB} - n_{A,B}) = -(\bar{e}+\bar{d})(\bar{b}-\bar{a}) - \bar{e}(\bar{c}) \le 0$ 0 because $\bar{b} \ge \bar{a}$. Thus $m_{AB,AB} + m_{A,AB} \ge 0$. A parallel argument shows that $m_{B,AB} + m_{AB,AB} \le 0$.

(5) We must show that $-m_{A,A} \ge m_{A,AB} \ge -m_{B,A}$. First, $-m_{A,A} - m_{A,AB} = n_{B,B} n_{AB,AB} - n_{B,AB} n_{B,AB} + n_{A,B} n_{B,AB} - n_{B,B} n_{A,AB} = n_{B,B} (n_{AB,AB} - n_{A,AB}) + n_{B,AB} (n_{A,B} - n_{B,AB}) = (\bar{e}+\bar{d})(\bar{c}) + (\bar{e}+\bar{c})(-\bar{c}) = \bar{c}(\bar{d}-\bar{c}) \ge 0$, because $\bar{d} \ge \bar{c}$. Thus $-m_{A,A} \ge m_{A,AB}$. Second, $m_{A,AB} + m_{A,B} = (n_{B,B} n_{A,AB} - n_{A,B} n_{B,AB} + n_{A,B} n_{AB,AB} - n_{A,AB} n_{B,AB})$. But this second term is equal to $\bar{e}(\bar{d}-\bar{c}) + \bar{e}\bar{a} + \bar{a}(\bar{d}-\bar{c})$, all components of which are nonnegative. Thus $m_{A,AB} + m_{A,B} \ge 0$, i.e., $m_{A,AB} \ge - m_{A,B}$.

(6) This proof is identical to that of (5). [X]

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